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Homogenization of the heat equation in a noncylindrical domain with randomly oscillating boundary

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In this article, we study the homogenization of heat equations in a domain with randomly oscillating boundary parts. The random oscillating boundary is time-dependent and confined by a stationary random field. Here, we follow a new homogenization technique that deals with the evolving domains, which covers many applications. We obtain the asymptotic limit as $\varepsilon \to 0$ in the reference configuration, in which the heat equation becomes a parabolic equation with random oscillating coefficients in the reference domain. To the best of our knowledge, this is the first result of the homogenization of problems on the random evolving boundary domain. One of the major contributions is the corrector result which we establish in this article.

KEYWORDS

evolving boundary, homogenization, parabolic equations, random oscillating boundary

MSC CLASSIFICATION

35B27; 35K05; 35R37; 35R60; 35Q74

1 | INTRODUCTION

We are interested in studying the problems on random micro heterogeneous media (the rough boundary) via a realization of a statistically homogeneous (spatial) process. We consider a heat equation on a noncylindrical oscillating boundary domain with a random oscillating boundary, which will be described later. Our aim is to study the existence, uniqueness, and homogenization of the following equation:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}(\hat{x}, x_n, t, \omega) - \Delta u_\varepsilon(\hat{x}, x_n, t, \omega) = f(\hat{x}, x_n, t) & \text{in } Q_\varepsilon^T, \\ u_\varepsilon(\hat{x}, \eta_{\omega,\varepsilon}(\hat{x}, t), t, \omega) = \frac{\partial \eta_\varepsilon}{\partial t}(\hat{x}, t, \omega) & \text{on } \Sigma_\varepsilon^T, \\ u_\varepsilon(\hat{x}, 0, t, \omega) = 0 & \text{on } \Gamma_0 \times (0, T), \\ u_\varepsilon(\hat{x}, x_n, 0, \omega) = h(\hat{x}, x_n) & \text{in } \Omega_\varepsilon^0, \\ \hat{x} \mapsto u_\varepsilon(\hat{x}, x_n, t) & \text{is } (0,1)^{n-1}\text{-periodic.} \end{cases} \tag{1.1}$$

A similar problem was studied in Muthukumar et al,[1] where it was assumed that the microstructure on the evolving boundary is periodic, but here, we consider random microstructure. However, the effective problem depends on the cell problem in random space, but here, we choose some particular test functions to overcome the dependence of random space on the homogenization process.

Many mathematical models represented by PDEs on time-dependent domains (noncylindrical domains) are important in real-life applications. Artificial hearts and automobile engines are examples of devices that work with a fluid. Recently, many researchers give attention to the problems on noncylindrical domains with more general data and equations. The lack of uniform bound in a fixed function space for the homogenization problems on the noncylindrical domain is solved by transforming the problem on the reference domain.

Recently, many mathematical methods have been devoted to study the homogenization of PDEs with microstructures due to the applications in material sciences and engineering, fluid flow over channels with rough boundaries, and so on. The engineer's strategy to avoid an expensive numerical computation of the solutions of the problem with microstructures is to find the effective boundary condition for the effective (homogenized) problem on a nice domain with the absence of microstructure. These observations lead engineers to imitate this effect in practical applications such as geophysical fluid dynamics,[2] drag reduction in aviation and biological structures,[3–5] fluid–structure interaction problems, and many more scientific fields. The periodic homogenization of linear problems is a well-understood subject (cf. previous studies[6–10] and reference therein). Significant works are carried out for the homogenization of problems on highly oscillating boundaries; we refer to the reader (cf. other works[11–19]).

The work in Achdou et al[20] deals with the homogenization of steady-state incompressible Navier–Stokes equation (Laminar flow) in a domain with the periodic oscillating boundary of order $O(\varepsilon)$ amplitude. The elliptic equation with highly oscillating coefficients in an oscillating boundary domain of small amplitude oscillations considered in Allaire and Amar.[21] They used the boundary layer techniques to derive the effective problem and the order of convergence. The Navier–Stokes system on a cylindrical domain with an oscillating boundary of amplitude $O(\varepsilon)$ was studied in Bucur et al.[16]

There are many results on the boundary value problems on oscillating domains with a fixed amplitude, that is, the amplitude of $O(1)$. Brizzi and Chalot[22] studied the homogenization of the Laplace equation with homogeneous Neumann boundary conditions in domains with the periodic oscillating boundary of fixed amplitude. The inhomogeneous Neumann boundary condition was considered in Gaudiello.[23] For further works on homogenization of similar problems, we refer to Amirat et al. and Gaudiello and Guibé.[15,24] The asymptotic analysis of optimal control problems was considered in other works.[25–27] In Aiyappan et al,[12] the generalized unfolding method is used to derive the homogenization problem on oscillating circular boundary domain with amplitude of $O(1)$.

We refer the reader to the monograph[28] for an excellent introduction to random homogenization problems and their applications. The classical approach to proving the homogenization result for the second-order elliptic equation with random coefficients consists of solving the auxiliary problem on random space and then identify the effective coefficients. The first homogenization result in the stationary ergodic random elliptic and parabolic operators was obtained in the pioneering works.[29,30] An extensive literature has been developed later on to obtain a rigorous derivation for the homogenization of stochastic models (cf. previous works[31–36]). In Bourgeat et al.,[37] *two-scale convergence in the mean* was introduced to study the random problems. They used the ergodic theory techniques to obtain the homogenized problem. For further works on stochastic two-scale convergence, we refer to the previous articles.[38,39] For the homogenization of the parabolic problems with periodic coefficients in space variable and stationary random in time variable, we refer to other studies[40–42] and reference therein.

The random boundary homogenization was studied in Amirat et al[13] for the Laplace equation with the Fourier boundary condition, and the random oscillating boundary is given by the random perturbation of a fixed hyper-surface. We also mention Chechkin et al.'s paper,[43] where the boundary oscillations have both periodic and random structures and inhomogeneous Fourier boundary condition taken into account for the Poisson equation. Peter[44] studied the homogenization of a coupled parabolic equations in a domain (which contains evolving two-phase medium) by transformation into fixed periodic domain (reference domain). The interfacial exchange between reaction and diffusion problems with homogeneous Neumann boundary conditions is described in the micro problem. In last few years, many authors have studied the homogenization problems in domains with evolving interface (cf. other works[45–51] and the references therein).

Motivated by the Muthukumar et al.'s work,[1] the aim of this article is to present the homogenization procedure of the problem (1.1). However, the combination of the oscillating boundary and its complex geometry, our problem leads to additional mathematical difficulties to deal with. To our knowledge, there are no results on the homogenization of the problems on randomly evolving domains. Considering the random evolving oscillating boundary is itself the novelty of the present work. On the other hand, we also obtained the effective problem without solving the auxiliary problem in random space. Another important fact for the numerical approximation is that this approach unified the small oscillating boundary problems (even time-independent) into the oscillating coefficient problems.

The article is organized as follows. In Section 2, the problem descriptions are presented. We recall the preliminaries of random structures in Section 3. In Section 4, we discuss the existence and uniqueness with a uniform estimate of solutions to the problems in the reference domain. Further, we obtain the convergence of data. The various convergence of solutions and the effective problem on the reference domain are established in Section 5. The energy convergence of problems and the corrector results in the appropriate norms (strong convergence) are shown in Section 6.

2 | PROBLEM DESCRIPTION

To describe random microstructure, let $(\mathcal{O},\mathcal{E},\mathbb{P})$ be a probability space with an ergodic $n-1$ dynamical system $\tau_{\hat{x}}, \hat{x} \in \mathbb{R}^{n-1}$ (q.v Definition 3.1). Let $L^p(\mathcal{O},\mathcal{E},\mathbb{P})$, $1 \le p < \infty$ be the space of equivalence classes of $\mathbb{P}$-measurable functions and $\mathbb{P}$-integrable with exponent p. Let $L^\infty(\mathcal{O})$ be the space of $\mathbb{P}$-a.s essentially bounded functions. For simplicity, we write $L^p(\mathcal{O})$ instead of $L^p(\mathcal{O},\mathcal{E},\mathbb{P})$ in the rest of the article. Let us assume that $\mathcal{E}$ is countably generated, so that the spaces $L^p(\mathcal{O})$ with $p\in[1,\infty)$ are separable. We refer the reader to the next section for more about randomness. Let $\boldsymbol{\eta},\boldsymbol{\eta}_1 : \mathcal{O}\times[0,T]\to\mathbb{R}$ be random functions such that $3/2 \le \boldsymbol{\eta} \le 2$ and $-1 \le \boldsymbol{\eta}_1 \le 0$. Let us define $\eta,\eta_1 : \mathcal{O}\times\mathbb{R}^{n-1}\times[0,T]\to\mathbb{R}$ by $\eta(\omega,\hat{x},t) = \boldsymbol{\eta}(\tau_{\hat{x}}\omega,t)$, $\eta_1(\omega,\hat{y},t) = \boldsymbol{\eta}_1(\tau_{\hat{y}}\omega,t)$. For each $\omega\in\mathcal{O}$, the functions $\eta_\omega,\eta_{1,\omega} : \mathbb{R}^{n-1}\times[0,T]\to\mathbb{R}$ defined by $\eta_\omega(\hat{x},t) := \eta(\omega,\hat{x},t)$, $\eta_{1,\omega}(\hat{y},t) := \eta_1(\omega,\hat{y},t)$. For each $\omega\in\mathcal{O}$, the functions η_ω and $\eta_{1,\omega}$ are realizations of $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_1$, respectively.

Assume that $0<\varepsilon\ll 1$ with $\varepsilon=\frac{1}{N}$, $N\in\mathbb{N}$ is a small parameter so that $\varepsilon\to 0$. Now, let us scale η_1 with parameter ε by

$$\eta_{1,\omega,\varepsilon}(\hat{x},t) = \eta_1\left(\omega,\frac{\hat{x}}{\varepsilon},t\right) = \boldsymbol{\eta}_1(\tau_{\frac{\hat{x}}{\varepsilon}}\omega,t). \tag{2.1}$$

The oscillating random variable $\eta_{\omega,\varepsilon}$ is defined by

$$\eta_{\omega,\varepsilon}(\hat{x},t) := \eta_\omega(\hat{x},t) + \varepsilon\eta_{1,\omega,\varepsilon}(\hat{x},t) \tag{2.2}$$

where the last term in the right hand side represents the random oscillations of the boundary. As an example, we consider simplest case of a “black-gray” checkerboard where the tiles can be only one of two possibilities. Consider the corresponding discrete sample space $\tilde{\mathcal{O}} = \{black, gray\}$, and define the measure $\tilde{\mu}$ on the measurable space $(\tilde{\mathcal{O}},2^{\tilde{\mathcal{O}}})$ as $\tilde{\mu}(black)=p$, $\tilde{\mu}(gray)=1-p$ with $p\in(0,1)$. The function $\eta_{1,\tilde{w}}t : \mathbb{T}\times[0,T]\to\mathbb{R}$ for each $\tilde{\omega}\in\tilde{\mathcal{O}}$ is defined as

$$\eta_{1,\tilde{w}}(\hat{y},t) = \begin{cases} \tilde{\eta}_1(\hat{y},t) & \text{if } \tilde{\omega} = black \\ \tilde{\eta}_2(\hat{y},t) & \text{if } \tilde{\omega} = gray, \end{cases}$$

where $\tilde{\eta}_i : [0,1]\times[0,T]\to[-1,0]$ $i=1,2$ be a smooth function such that $\tilde{\eta}_i(0,t)=\tilde{\eta}_j(1,t)$ for all $t\in[0,T]$, and extended periodically to $\mathbb{R}$. For this case, the probability space $(\mathcal{O},\mathcal{E},\mathbb{P})$ defined by $\mathcal{O} := \widehat{\mathcal{O}}\times\mathbb{T}$, $\mathcal{E}=\widehat{\mathcal{E}}\otimes\mathcal{B}(\mathbb{T})$, $\mathbb{P}=\mu\otimes\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure and the space $(\widehat{\mathcal{O}},\widehat{\mathcal{E}},\mu) = \Pi_{\mathbb{Z}}(\tilde{\mathcal{O}},2^{\tilde{\mathcal{O}}},\tilde{\mu})$ endowed with the product measure $\mu=\otimes_{\mathbb{Z}}\tilde{\mu}$. It remains to define measure preserving dynamical system on the constructed probability space. For this aim, we make use of the product space and corresponding dynamical systems. Let us denote the element $\hat{\omega}$ of $\widehat{\mathcal{O}}$ by the sequence $\hat{\omega}=\left\{\tilde{\omega}_j\right\}_{j\in\mathbb{Z}}$. Define the discrete dynamical system $\hat{\tau}=\left\{\hat{\tau}_k : \widehat{\mathcal{O}}\to\widehat{\mathcal{O}}\right\}_{k\in\mathbb{Z}}$ by $\hat{\tau}_k(\hat{\omega}) := \hat{\tau}_k(\left\{\tilde{\omega}_j\right\}_{j\in\mathbb{Z}}) = \left\{\tilde{\omega}_{j+k}\right\}_{j\in\mathbb{Z}}$. For each $\hat{y}\in\mathbb{R}$, we have a unique decomposition $\hat{y}=\lfloor\hat{y}\rfloor+(\hat{y}-\lfloor\hat{y}\rfloor)$, where $\lfloor\hat{y}\rfloor\in\mathbb{Z}$ is the greatest integer less or equal to $\hat{y}$ and $\hat{y}-\lfloor\hat{y}\rfloor\in\mathbb{T}$. Next, we define the standard dynamical system $\kappa=\left\{\kappa_{\hat{y}}\right\}_{\hat{y}\in\mathbb{R}}$ on $\mathbb{T}$ by $\kappa_{\hat{y}\in\mathbb{R}}(\nu)=\hat{y}+\nu-\lfloor\hat{y}+\nu\rfloor$. It is well-known that the dynamical systems $\hat{\tau}$ and κ are measure preserving ergodic dynamical systems on $\widehat{\mathcal{O}}$ and $\mathbb{T}$, respectively. Finally, we define the dynamical system $\tau=\left\{\tau_{\hat{y}}\right\}_{\hat{y}\in\mathbb{R}}$ on $\mathcal{O}$ by

$$\tau_{\hat{y}}(\omega) = \tau_{\hat{y}}(\hat{\omega},\nu) = \left(\hat{\tau}_{\lfloor\hat{y}+\nu\rfloor}(\hat{\omega}),\kappa_{\hat{y}}(\nu)\right),\ \hat{\omega}\in\widehat{\mathcal{O}},\ \nu\in\mathbb{T}.$$

One can easily verify that τ defines the dynamical system on $\widehat{\mathcal{O}}\times\mathbb{T}$. It is also known that the above dynamical system τ is ergodic. Now, we define the random variable $\boldsymbol{\eta}_1 : \mathcal{O}\times[0,T]\to\mathbb{R}$ by $\boldsymbol{\eta}_1(\omega,t) := \eta_{1,\tilde{\omega}_0}(\nu,t)$ for $\omega=(\left\{\tilde{\omega}_j\right\},\nu)\in\mathcal{O}$. Consequently, $\boldsymbol{\eta}_1(\tau_{\hat{y}}\omega,t) := \eta_{1,\tilde{\omega}_{\lfloor\hat{y}+\nu\rfloor}}(\hat{y}+\nu-\lfloor\hat{y}+\nu\rfloor,t)$. Similarly, one can define $\boldsymbol{\eta}$ on the probability space $\mathcal{O}$. For simplicity, we may take $\boldsymbol{\eta}$ to be deterministic function (for example the deterministic function $\boldsymbol{\eta}=6+\frac{\hat{x}}{8}\sin(2\pi\hat{x})$). A typical example of this situation is given by two different periodic functions (upto translation and scaling) continuously glued together (q.v. Figure 1),

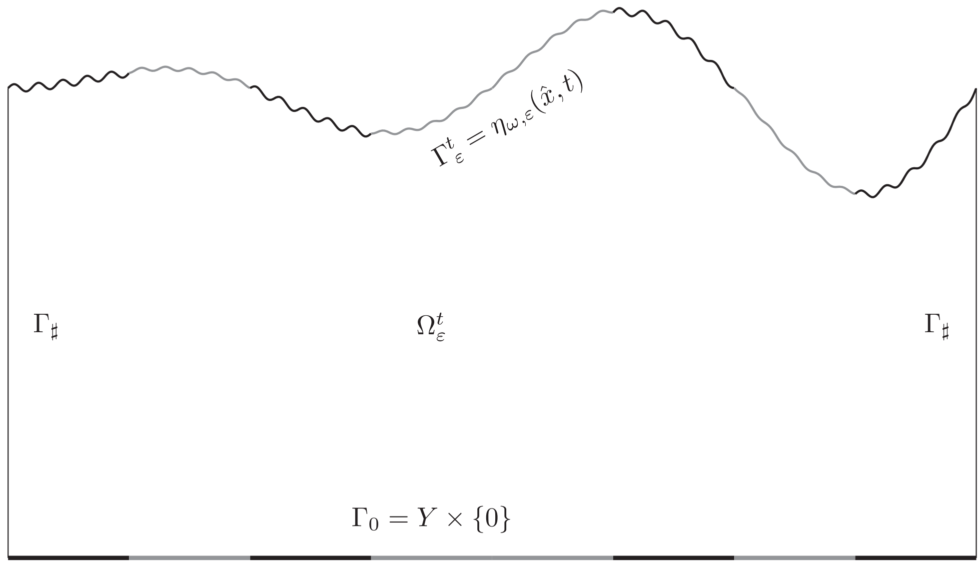

FIGURE 1 Slice of the evolving domain with oscillating boundary at time t

$$\eta_{1,\tilde{w}}(\hat{y},t) = \begin{cases} \tilde{\eta}_1(\hat{y},t) : \frac{\sin(2\pi\hat{y})-1}{2}\psi(t) & \text{if } \tilde{\omega} = black \\ \tilde{\eta}_2(\hat{y},t) := \frac{-1-\cos(2\pi\hat{y})}{4}\psi(t) & \text{if } \tilde{\omega} = gray, \end{cases}$$

where $\psi : [0,T] \to \mathbb{R}$ is smooth function and $\psi(t) \geq 1$ for all $t \in [0,T]$.

The σ-algebras $\hat{\mathcal{E}}$ and $\mathcal{B}(\mathbb{T})$ are countably generated, and therefore, the corresponding space $L^p(\mathcal{O})$ is separable. In most of the modeling of the natural phenomenon, the σ-field on probability space $\mathcal{O}$ is countably generated. We assumed the countably generated σ-field to avoid working with typical elements and typical trajectory for Birkhoff Ergodic Theorem.

For an another example, let $\mathcal{O} = \mathbb{T}^2$ the 2-dimensional torus, the Lebesgue measurable sets as the σ-algebra $\mathcal{E}$, and the probability measure $\mathbb{P}$ be the Lebesgue measure on $\mathbb{T}^2$. To define the dynamical system $\tau_{\hat{y}}$, let us fix an (2×2)-matrix $\Lambda = \left(\Lambda_{ij}\right)$. The dynamical system we define in the following way:

$$\tau_{\hat{y}}\omega = \omega + \Lambda\hat{y} \;(\text{mod} \;\; \mathbf{1}).$$

Obviously, the dynamical system $\tau = \left\{\tau_{\hat{y}}\right\}_{\hat{y}\in\mathbb{R}^2}$ preserves the measure $\mathbb{P}$ on $\mathcal{O}$. It is known that the dynamical system τ is ergodic if and only if $\Lambda_{ij}k_j \neq 0$ for any $k \in \mathbb{Z}^2, k \neq 0$. Since any measurable functions $\boldsymbol{\eta}_1$ on $\mathcal{O}$ can be identified with unique measurable 1-periodic function on $\mathbb{R}^2$, the realizations have the form $\boldsymbol{\eta}_1(\omega + \Lambda\hat{y})$. Let $\boldsymbol{\eta}_1$ be a continuous function on $\mathcal{O}$ then the corresponding realizations are called quasi-periodic functions. Again, let us take constant function $\boldsymbol{\eta} = 3/2$, and the diagonal matrix $\Lambda = diag(1, \sqrt{2})$. Let us choose the random function $\boldsymbol{\eta}_1(\omega_1,\omega_2,t) := (1+t)\frac{\sin(2\pi\omega_1)-\cos(2\pi\omega_2)-2}{4}$, and hence, the corresponding realizations are given by

$$\boldsymbol{\eta}_1(\omega + \Lambda\hat{y}, t) := (1+t)\frac{\sin(2\pi(\omega_1+\hat{y}_1)) - \cos(2\pi(\omega_2+\sqrt{2}\hat{y}_2)) - 2}{4}.$$

For each fixed $\varepsilon \in (0,1)$ and $t \in [0,T)$, let Ω_ε^t be defined by

$$\Omega_\varepsilon^t := \left\{(\hat{x},x_n) \in \mathbb{R}^n \,|\, \hat{x} \in \mathbb{T}^{n-1}, 0 < x_n < \eta_{\omega,\varepsilon}(\hat{x},t)\right\}$$

and the random oscillating part of the boundary of Ω_ε^t is defined by

$$\Gamma_\varepsilon^t := \left\{(\hat{x},\eta_{\omega,\varepsilon}(\hat{x},t)) \in \mathbb{R}^n \,|\, \hat{x} \in \mathbb{T}^{n-1}\right\}.$$

Since we have taken the $\hat{x}$ from the $(n-1)$ dimensional torus $\mathbb{T}^{n-1}$, which is equivalent to considering the $Y := (0,1)^{n-1}$-periodicity condition on the lateral boundaries of Ω_ε^t. The lower part of the boundary of Ω_ε^t (q.v. Figure 1) is denoted by

$$\Gamma_0 := \left\{(\hat{x}, 0) \in \mathbb{R}^n \,|\, \hat{x} \in (0,1)^{n-1}\right\}.$$

Let $Q_\varepsilon^T := \cup_{t\in(0,T)}\Omega_\varepsilon^t \times \{t\}$ be a noncylindrical (time-dependent) domain with an oscillating random boundary $\Sigma_\varepsilon^T := \cup_{t\in(0,T)}\Gamma_\varepsilon^t \times \{t\}$.

Let $\{\varepsilon\} \subset (0,1)$ be a given sequence of real numbers converging to zero such that $1/\varepsilon \in \mathbb{N}$. We make the following additional hypotheses on both $\boldsymbol{\eta}$ and $\boldsymbol{\eta}o$:

(H1) $\partial_\omega \boldsymbol{\eta}, \partial_\omega \boldsymbol{\eta}_1 \in L^\infty(\mathcal{O}\times(0,T))^{n-1}$ and $\frac{\partial \boldsymbol{\eta}}{\partial t}, \frac{\partial \boldsymbol{\eta}_1}{\partial t} \in L^\infty(\mathcal{O}\times(0,T))$,

where $L^\infty(\mathcal{O}\times(0,T))$ is the space of essentially bounded function with respect to the product σ-algebra of $\mathcal{E}$ and the Lebesgue measurable sets of $(0,T)$.

For each $0 \le t < T$, let

$$\hat{\Omega}^t := \left\{(\hat{x}, x_n) \in \mathbb{R}^n \,|\, \hat{x} \in \mathbb{T}^{n-1}, 0 < x_n < \max_{(\hat{x},\hat{y},\omega)\in\mathbb{T}^{n-1}\times\mathbb{T}^{n-1}\times\mathcal{O}} [\eta_\omega(\hat{x},t) + \eta_{1,\omega}(\hat{y},t)]\right\}$$

be the domain with a non-oscillating boundary and containing Ω_ε^t for all ε. Further, let $\hat{Q}^T := \cup_{t\in(0,T)}\hat{\Omega}_t \times \{t\}$ be the larger noncylindrical space-time domain containing Q_ε^T for all ε. Given $f \in L^2(\hat{Q}^T)$ and $h \in L^2(\hat{\Omega}_0)$, we wish to study the existence, uniqueness and homogenization of (1.1).

3 | PRELIMINARIES OF RANDOM STRUCTURES

Definition 3.1. The family of bijective measurable maps $\tau = \{\tau_{\hat{x}} : \mathcal{O} \to \mathcal{O}\}_{\hat{x}\in\mathbb{R}^{n-1}}$ is called a $n-1$ dynamical system on $\mathcal{O}$ if

(a) $\tau_0 = \mathrm{Id}$, and $\tau_{\hat{x}+\hat{x}_1} = \tau_{\hat{x}} \circ \tau_{\hat{x}_1}$, $\quad \hat{x}, \hat{x}_1 \in \mathbb{R}^{n-1}$;
(b) $\mathbb{P}(\tau_{\hat{x}}(E)) = \mathbb{P}(E) \quad \forall \hat{x} \in \mathbb{R}^{n-1},\ E \in \mathcal{E}$;
(c) The map $\tilde{\tau} : \mathbb{R}^{n-1}\times\mathcal{O} \to \mathcal{O}$, $(\hat{x}, \omega) \mapsto \tau_{\hat{x}}(\omega)$ is measurable with respect to product σ algebra, where $\mathbb{R}^{n-1}$ equipped with the Borel σ-algebra.

Definition 3.2.

(i) A measurable function $\phi(\omega)$ in $\mathcal{O}$ is said to be τ invariant if $\phi(\tau_{\hat{x}}(\omega)) = \phi(\omega)$ for any $\hat{x} \in \mathbb{R}^{n-1}$ and almost all $\omega \in \mathcal{O}$.
(ii) The dynamical system τ is called ergodic if

$$\mathbb{P}(E) = 0 \text{ or } 1 \qquad \forall E \subset \mathcal{O} \text{ with } \tau_{\hat{x}}E = E\ \forall \hat{x} \in \mathbb{R}^{n-1}.$$

Equivalently, in terms of invariant functions, *a dynamical system τ is ergodic* if every τ-invariant function is constant almost everywhere, that is,
$g(\tau_{\hat{x}}(\omega)) = g(\omega)$ for all $\hat{x}$ and almost all ω implies that $g \equiv$ constant $\mathbb{P}$-a.s.
(iii) A random variable $\nu : \mathbb{R}^{n-1}\times\mathcal{O} \to \mathbb{R}$ is said to be statistically homogeneous if there exists a random variable $\tilde{\nu} : \mathcal{O} \to \mathbb{R}$ such that $\nu(\hat{x},\omega) = \tilde{\nu}(\tau_{\hat{x}}\omega)$, where $\tau_{\hat{x}}$ is a $n-1$ dynamical system.
(iv) For any $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $d \in \mathbb{N}$, the mean value (spatial average) of φ denoted by $\mathcal{M}_\varphi$ and is defined as

$$\mathcal{M}_\varphi := \lim_{\varepsilon\to 0} \frac{1}{|B|}\int_B \varphi\left(\frac{x}{\varepsilon}\right) dx$$

for all bounded Lebesgue measurable sets $B \subset \mathbb{R}^d$.
(v) The mathematical expectation associated to the probability space $(\mathcal{O}, \mathcal{E}, \mathbb{P})$ is denoted by $\mathbb{E}$ and is defined by

$$\mathbb{E}(g) = \int_{\mathcal{O}} g\, d\mathbb{P} \qquad \text{for} \quad g \in L^1(\mathcal{O}).$$

The following result holds true, and we refer to Jikov et a.[28, Section 7.1] for the proof:

Lemma 3.1. *Let $(\mathcal{O}, \mathcal{E}, \mathbb{P})$ be a probability space, and let $\tau = \{\tau_{\hat{x}}\}_{\hat{x}\in\mathbb{R}^{n-1}}$ be a $(n-1)$-dynamical system on $\mathcal{O}$.*

(a) Let $g \in L^p(\mathcal{O})$ for $p \geq 1$, then almost all $\omega \in \mathcal{O}$ the realizations $g(\tau_{(\cdot)}\omega)$ belong to $L^p_{loc}\left(\mathbb{R}^{n-1}\right)$.
(b) Let $\{g_k\}_{k=1}^{\infty} \subset L^p(\mathcal{O})$ and $g \in L^p(\mathcal{O})$ such that $g_k \to g$ in $L^p(\mathcal{O})$ as $k \to \infty$. Then there exists a subsequence $\left\{k_j\right\}_{j=1}^{\infty}$ of $\{k\}$ such that almost all $\omega \in \mathcal{O}$

$$\lim_{j\to\infty} \left\| g_{k_j}(\tau_{(\cdot)}\omega) - g(\tau_{(\cdot)}\omega)\right\|_{L^p_{loc}(\mathbb{R}^{n-1})} = 0.$$

Using the natural action on $L^2(\mathcal{O})$, we define $(n-1)$-parameter group of operators $\mathcal{U}_{\hat{x}} : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by $\mathcal{U}_{\hat{x}} g = g \circ \tau_{\hat{x}}$. Thanks to the conditions of the Definition 3.1, the action is unitary for each $\hat{x} \in \mathbb{R}^{n-1}$ and are strongly continuous for all $\hat{x} \in \mathbb{R}^{n-1}$, that is, $\lim_{\hat{x}\to 0}\|\mathcal{U}_{\hat{x}} g - g\|_{L^2(\mathcal{O})} = 0$ for all $g \in L^2(\mathcal{O})$. We next define the infinitesimal generators ∂^j_ω, $j = 1, 2, \ldots, n-1$ associated with the unitary group $\mathcal{U}_{\hat{x}\cdot \boldsymbol{e}_j}$ by

$$\partial^j_\omega g(\cdot) = \lim_{\substack{x_j\neq 0, x_j\to 0\\ x_k=0, k\neq j}} \frac{g(\tau_{x_j}\cdot) - g(\cdot)}{x_j} \quad \text{in} \quad L^2(\mathcal{O}).$$

For each $1 \leq j \leq n-1$, $\mathcal{D}_j(\mathcal{O})$ denotes the domain of the operator ∂^j_ω. Consequently, the operators ∂^j_ω are skew-symmetric, that is,

$$\int_{\mathcal{O}} g_1 \partial^j_\omega g \, d\mathbb{P} = -\int_{\mathcal{O}} g \partial^j_\omega g_1 \, d\mathbb{P}, \text{ for all } \ g, g_1 \in \mathcal{D}_j(\mathcal{O}).$$

In particular, $\int_{\mathcal{O}} \partial^j_\omega g \, d\mathbb{P} = 0$, $j = 1, \ldots, n-1$. Now, we define $W^{1,2}(\mathcal{O})$ by the domain of the operator $\nabla_\omega := \left(\partial^1_\omega, \ldots, \partial^{n-1}_\omega\right)$, that is,

$$W^{1,2}(\mathcal{O}) := \mathcal{D}_1(\mathcal{O}) \cap \ldots \cap \mathcal{D}_{n-1}(\mathcal{O}) = \left\{ g \in L^2(\mathcal{O}) : \partial^j_\omega g \in L^2(\mathcal{O}), \ 1 \leq j \leq n-1 \right\},$$

with the natural graph norm. Further, for almost all ω, we have

$$\frac{\partial v}{\partial x_j}(\hat{x}, \omega) = \partial^j_\omega g(\tau_{\hat{x}}\omega) \text{ and } \nabla_{\hat{x}} v(\hat{x}, \omega) = \nabla_\omega g(\tau_{\hat{x}}\omega),$$

where $v(\hat{x}, \omega) := g(\tau_{\hat{x}}\omega)$ is the realization of g. For each $k \in \mathbb{N}$, we can define

$$W^{k,2}(\mathcal{O}) = \left\{ g \in L^2(\mathcal{O}) : \left(\partial^1_\omega\right)^{\alpha_1} \ldots \left(\partial^{n-1}_\omega\right)^{\alpha_n} g \in L^2(\mathcal{O}), \ \alpha_1 + \ldots + \alpha_{n-1} = k \right\}.$$

Let us also define the spaces $W^{\infty,2}(\mathcal{O}) = \bigcap_{k\in\mathbb{N}} W^{k,2}(\mathcal{O})$ and

$$C^\infty(\mathcal{O}) = \left\{ g \in W^{\infty,2}(\mathcal{O}) : \forall (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0, \ \left(\partial^1_\omega\right)^{\alpha_1} \ldots \left(\partial^{n-1}_\omega\right)^{\alpha_n} g \in L^\infty(\mathcal{O}) \right\}.$$

We remark here that $C^\infty(\mathcal{O})$ is the stochastic analog of the smooth functions. Moreover, it can be shown that the space $C^\infty(\mathcal{O})$ is dense in $L^p(\mathcal{O})$ and in $W^{1,p}(\mathcal{O})$, $1 \leq p < \infty$. For the above terminology, we refer to the monographs.[28,52] The next result (Birkhoff theorem) is an important result in ergodic theory,[28, Theorem 7.2] and it plays a pivotal role in homogenization of random PDEs.

Theorem 3.1 (Birkhoff Ergodic Theorem). *Let $(\mathcal{O}, \mathcal{E}, \mathbb{P})$ be a probability space, and let $\tau = \{\tau_{\hat{x}}\}_{\hat{x}\in\mathbb{R}^{n-1}}$ be a $(n-1)$-dynamical system on $\mathcal{O}$. For each $\varphi \in L^p(\mathcal{O})$ with $p \geq 1$, let $\varphi^\varepsilon_\tau(\hat{x}, \omega) = \varphi(\tau_{\frac{\hat{x}}{\varepsilon}}\omega)$ for all $\varepsilon > 0$. Then*

(i) the realization $\varphi^\varepsilon_\tau(\hat{x}, \omega)$ satisfies

$$\varphi^\varepsilon_\tau(\cdot, \omega) \rightharpoonup \mathcal{M}_\varphi(\omega) \ \text{in} \ \ L^p_{loc}(\mathbb{R}^{n-1}), \text{ as } \ \varepsilon \to 0,$$

for almost all $\omega \in \mathcal{O}$.

(ii) $\mathcal{M}_\varphi$ is a τ-invariant function, that is, $\mathcal{M}_\varphi\left(\tau_{\hat{x}}(\omega)\right) = \mathcal{M}_\varphi(\omega)$ for all $\hat{x} \in \mathbb{R}^n$, $\mathbb{P}$-a.s. Also,

$$\mathbb{E}(\varphi) := \int_{\mathcal{O}} \varphi(\omega) \, d\mathbb{P} = \int_{\mathcal{O}} \mathcal{M}_\varphi(\omega) \, d\mathbb{P} = \mathbb{E}(\mathcal{M}_\varphi).$$

(iii) *In addition, suppose the dynamical system τ is ergodic, we have*

$$\varphi_\tau^\varepsilon(\cdot,\omega) \rightharpoonup \int_{\mathcal{O}} \varphi(\omega)\, d\mathbb{P} \ \ in \ \ L^p_{loc}(\mathbb{R}^{n-1}), \ as \ \ \varepsilon \to 0,$$

for almost all $\omega \in \mathcal{O}$.

We remark that the Birkhoff's theorem yields the mean value $\mathcal{M}_\varphi$ is constant almost everywhere and is given by $\mathcal{M}_\varphi = \mathbb{E}(\varphi)$ whenever the dynamical system τ is ergodic. For each $1 \le i \le n-1$, the hypothesis (H1) and Theorem 3.1(i) yields

$$\frac{\partial \eta_{1,\omega}}{\partial y_i}\left(\frac{\hat{x}}{\varepsilon}, t\right) \rightharpoonup \mathcal{M}_{\frac{\partial \eta_{1,\omega}}{\partial y_i}}(\omega, t) = \int_{\mathbb{T}^{n-1}} \frac{\partial \eta_1}{\partial y_i}(\hat{y}, \omega, t)\, d\hat{y} = 0, \tag{3.1}$$

where the last equality follows from the integration by parts.

4 | EXISTENCE AND UNIQUENESS RESULT

In this section, we will use a coordinate transformation for u_ε to obtain existence and uniqueness and hence uniform bounds of solutions in appropriate function spaces. This approach has been discussed in Muthukumar et al[1] for the deterministic heat equation and the effective coefficients in the reference domain using the two-scale convergence with appropriate modifications. To this end, let us introduce the coordinate transformation $\mathcal{T}_{\omega,\varepsilon} : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n \times [0,T)$ defined by $\mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, t) = (\hat{x}, z\eta_{\omega,\varepsilon}(\hat{x}, t), t)$. A consequence of hypothesis (H1), the transformation $\mathcal{T}_{\omega,\varepsilon}$ is a Lipschitz diffeomorphism. From the definition of $\mathcal{T}_{\omega,\varepsilon}$, we have $\mathcal{T}_{\omega,\varepsilon}(Q^T) = Q^T_\varepsilon$ where $Q^T = \Omega \times (0,T)$ is so called reference domain with

$$\Omega = \mathbb{T}^{n-1} \times (0,1), \Gamma_0 = \mathbb{T}^{n-1} \times \{0\}, \ \text{and} \ \ \Gamma_1 = \mathbb{T}^{n-1} \times \{1\}.$$

From now on, we denote by $x := (\hat{x}, z)$ the variable of the domain (time independent) Ω and the points in variable domain (time dependent) Ω^t_ε is denoted by $(\hat{x}, x_n)$. We now define the functions $U_\varepsilon, f_\varepsilon : Q^T \to \mathbb{R}$ and $h_\varepsilon : \Omega \to \mathbb{R}$ corresponding to u_ε, f, and h in the reference configuration, respectively, by

$$U_\varepsilon(\hat{x}, z, t) = u_\varepsilon \circ \mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, t), \ f_\varepsilon(\hat{x}, z, t) = f \circ \mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, t) \text{and} \ \ h_\varepsilon(\hat{x}, z) = h \circ \mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, 0).$$

In order to find the equation satisfied by U_ε in the reference domain, it is necessary to find the differential of $\mathcal{T}^{-1}_{\omega,\varepsilon}$. The differential of $\mathcal{T}^{-1}_{\omega,\varepsilon}$ is denoted by $D\mathcal{T}^{-1}_{\omega,\varepsilon}$ and is given by the square matrix

$$D\mathcal{T}^{-1}_{\omega,\varepsilon}(\hat{x}, x_n, t) = \begin{pmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{(n-1)\times 1} \\ -\frac{x_n}{\eta^2_{\omega,\varepsilon}(\hat{x},t)} \nabla^T_{\hat{x}} \eta_{\omega,\varepsilon}(\hat{x}, t) & \frac{1}{\eta_{\omega,\varepsilon}(\hat{x},t)} & -\frac{x_n}{\eta^2_{\omega,\varepsilon}(\hat{x},t)} \frac{\partial \eta_{\omega,\varepsilon}}{\partial t}(\hat{x}, t) \\ \mathbf{0}_{1\times(n-1)} & 0 & 1 \end{pmatrix}_{(n+1)\times(n+1)},$$

where I_{n-1} is the $(n-1)$ size identity matrix, and $\boldsymbol{0}_{(n-1)\times 1}$ is the $(n-1)$-tuple zero (column) vector. A consequence of hypothesis (H1) and the uniform bound of $\eta^{-k}_{\omega,\varepsilon}$, for $k = 1, 2$, is that the entries in $D\mathcal{T}^{-1}_{\omega,\varepsilon}$ belong to $L^\infty(\hat{Q}^T)$. Using the chain rule for composition of functions, the first derivatives of u_ε in terms of U_ε is given by

$$Du_\varepsilon(\hat{x}, x_n, t) = D(U_\varepsilon \circ \mathcal{T}^{-1}_{\omega,\varepsilon})(\hat{x}, x_n, t) = DU_\varepsilon \circ \mathcal{T}^{-1}_{\omega,\varepsilon}(\hat{x}, x_n, t) D\mathcal{T}^{-1}_{\omega,\varepsilon}(\hat{x}, x_n, t).$$

Hence, in the reference domain variables $(\hat{x}, z, t)$,

$$(Du_\varepsilon) \circ \mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, t) = DU_\varepsilon\, (D\mathcal{T}^{-1}_{\omega,\varepsilon}) \circ \mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, t).$$

Let us introduce the matrix $M_{\omega,\varepsilon} : Q^T \to \mathbb{R}^{n\times n}$ defined by

$$M_{\omega,\varepsilon}(\hat{x}, z, t, \omega) = \begin{pmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times 1} \\ -\frac{z}{\eta_{\omega,\varepsilon}(\hat{x},t)} (\nabla_{\hat{x}} \eta_{\omega,\varepsilon}(\hat{x}, t))^{tr} & \frac{1}{\eta_{\omega,\varepsilon}(\hat{x},t)} \end{pmatrix}_{n\times n}, \tag{4.1}$$

where the superscript tr denotes the transpose matrix. Under the coordinate transformation $\mathcal{T}_{\omega,\varepsilon}$, the operators ∇ and div transform as

$$(\nabla v)\circ \mathcal{T}_{\omega,\varepsilon} = M^{tr}_{\omega,\varepsilon}\nabla(v\circ\mathcal{T}_{\omega,\varepsilon}) \text{ and } \operatorname{div}(\boldsymbol{v})\circ\mathcal{T}_{\omega,\varepsilon} = M^{tr}_{\omega,\varepsilon}\nabla\cdot(\boldsymbol{v}\circ\mathcal{T}_{\omega,\varepsilon}),$$

respectively, where $v : \Omega^t_\varepsilon \to \mathbb{R}$ and $\boldsymbol{v}$ is the n-component of such functions. By a straightforward calculations, we get the identity $M^{tr}_{\omega,\varepsilon}\nabla\cdot\Phi = \eta^{-1}_{\omega,\varepsilon}\operatorname{div}\left(\eta_{\omega,\varepsilon}M_{\omega,\varepsilon}\Phi\right)$ for all $\Phi \in H^1_0(\Omega)^n$. Thus, Equation (1.1) in the new variable $(\hat{x}, z) \in \Omega$, in terms of U_ε, can be written as

$$\begin{cases} \frac{\partial U_\varepsilon}{\partial t} - \frac{1}{\eta_{\omega,\varepsilon}}\operatorname{div}(A_\varepsilon\nabla U_\varepsilon) - \frac{z}{\eta_{\omega,\varepsilon}}\frac{\partial\eta_{\omega,\varepsilon}}{\partial t}\frac{\partial U_\varepsilon}{\partial z} = f_\varepsilon & \text{in } Q^T, \\ U_\varepsilon = \frac{\partial\eta_{\omega,\varepsilon}}{\partial t} & \text{on } \Gamma_1\times(0,T), \\ U_\varepsilon = 0 & \text{on } \Gamma_0\times(0,T), \\ U_\varepsilon = h_\varepsilon & \text{in } \Omega, \end{cases} \tag{4.2}$$

where $A_\varepsilon := \eta_{\omega,\varepsilon}M_{\omega,\varepsilon}M^{tr}_{\omega,\varepsilon} = \begin{pmatrix} \eta_{\omega,\varepsilon}I_{n-1} & -z(\nabla_{\hat{x}}\eta_{\omega,\varepsilon}(\hat{x},t)) \\ -z(\nabla_{\hat{x}}\eta_{\omega,\varepsilon}(\hat{x},t))^T & \frac{1+z^2|\nabla_{\hat{x}}\eta_{\omega,\varepsilon}|^2}{\eta_{\omega,\varepsilon}} \end{pmatrix}$.

Since the existence and uniqueness of solution depends on the regularity of the data, we assume that the realizations as boundary data satisfy $\frac{\partial\eta_\omega}{\partial t}, \frac{\partial\eta_{1,\omega}}{\partial t} \in H^{\frac{1}{2}}(\mathbb{T}^{n-1}\times(0,T))$. For the homogenization, it is enough to consider homogeneous boundary condition on $\Gamma_1\times(0,T)$ due to the lifting of parabolic boundary data (cf. Muthukumar et al.[1]). We rewrite (4.2) in the following form with homogeneous boundary condition:

$$\begin{cases} \eta_{\omega,\varepsilon}\frac{\partial U_\varepsilon}{\partial t} - \operatorname{div}(A_\varepsilon\nabla U_\varepsilon) - z\frac{\partial\eta_{\omega,\varepsilon}}{\partial t}\frac{\partial U_\varepsilon}{\partial z} = \eta_{\omega,\varepsilon}f_\varepsilon & \text{in } Q^T, \\ U_\varepsilon = 0 & \text{on } (\Gamma_0\cup\Gamma_1)\times(0,T), \\ U_\varepsilon(\hat{x},z,0) = h_\varepsilon & \text{in } \Omega. \end{cases} \tag{4.3}$$

In the rest of the article, we consider the above equation. Indeed, the entries of A_ε are realization of random variable and time-dependent. Due to our hypothesis (H1) and the fact that $\{\eta^{-1}_{\omega,\varepsilon}\}$ is uniformly bounded in $\mathcal{O}\times\mathbb{T}^{n-1}\times[0,T))$, there exist $\beta > 0$ (independent of $\varepsilon, \omega, x, t$) such that, for all $\xi \in \mathbb{R}^n$, a.s

$$|A_\varepsilon(x,t,\omega)\xi| \le \beta|\xi| \qquad \text{a.e.}$$

The problem to be well-posed, we have to get uniform ellipticity of A_ε. This issue has been addressed in the recent article[1] for the deterministic case with a slightly different matrix, namely deterministic matrix $\frac{1}{\eta_\varepsilon}A_\varepsilon$. For the completeness, here, we show that A_ε is uniformly elliptic with the assumption that $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_1$ satisfy (H1). For any symmetric matrix $\Lambda = \begin{pmatrix} A & B \\ B^{tr} & C \end{pmatrix}$ with invertible matrix A, we have the following property: Λ is positive definite if and only if A is positive definite, and its Schur complement $C - B^{tr}A^{-1}B$ is positive definite. Since the matrix A_ε is symmetric, we can use a characterization of symmetric positive definite matrices with Schur complements. Using the Schur complement of $\eta_{\omega,\varepsilon}I_{n-1}$ in A_ε and the Aitken block-diagonalization formula for A_ε, we get $A_\varepsilon = N^{tr}_\varepsilon P_\varepsilon N_\varepsilon$, where

$$N_\varepsilon = \begin{pmatrix} I_{n-1} & -\frac{z}{\eta_{\omega,\varepsilon}}\nabla_{\hat{x}}\eta_{\omega,\varepsilon} \\ \boldsymbol{0}_{1\times(n-1)} & 1 \end{pmatrix}, \; N^{-1}_\varepsilon = \begin{pmatrix} I_{n-1} & \frac{z}{\eta_{\omega,\varepsilon}}\nabla_{\hat{x}}\eta_{\omega,\varepsilon} \\ \boldsymbol{0}_{1\times(n-1)} & 1 \end{pmatrix} \text{ and } P_\varepsilon = \begin{pmatrix} \eta_{\omega,\varepsilon}I_{n-1} & \boldsymbol{0}_{(n-1)\times1} \\ \boldsymbol{0}_{1\times(n-1)} & \frac{1}{\eta_{\omega,\varepsilon}} \end{pmatrix}.$$

Observe that $P_\varepsilon\xi\cdot\xi \ge \frac{1}{2}|\xi|^2$ for all $\xi \in \mathbb{R}^n$, and hence, P_ε is uniformly elliptic. An immediate consequence of hypothesis (H1) is that the norm $\|N^{-1}_\varepsilon\|^2 = n + \frac{z^2}{\eta^2_{\omega,\varepsilon}}|\nabla_{\hat{x}}\eta_{\omega,\varepsilon}|^2 \le n + 4\left\|\nabla_{\hat{x}}\eta_{\omega,\varepsilon}\right\|^2_{L^\infty(Q^T)} < \alpha_0$, where $\alpha_0 := n + 8\left[\|\partial_\omega\boldsymbol{\eta}\|^2_{L^\infty(\mathcal{O}\times(0,T))} + \|\partial_\omega\boldsymbol{\eta}_1\|^2_{L^\infty(\mathcal{O}\times(0,T))^{n-1}}\right]$ is independent of ε, t, ω. Therefore,

$$A_\varepsilon(\hat{x},z,t,\omega)\xi\cdot\xi = N^{tr}_\varepsilon P_\varepsilon N_\varepsilon\xi\cdot\xi = P_\varepsilon N_\varepsilon\xi\cdot N_\varepsilon\xi > \frac{1}{2}|N_\varepsilon\xi|^2 \ge \frac{1}{2\|N^{-1}_\varepsilon\|^2}|\xi|^2 > \frac{1}{2\alpha_0}|\xi|^2,$$

and hence, the existence of a elliptic constant $\alpha > 0$ for A_ε is guaranteed with $\alpha = 1/2\alpha_0$. So, we have shown that A_ε is uniformly elliptic with deterministic elliptic constant α. Let $H^1_0(\Omega)$ be the subspace in $H^1(\Omega)$ of functions vanishing on $\Gamma_0\cup\Gamma_1$, and $H^{-1}(\Omega)$ is the dual of $H^1_0(\Omega)$. We denote by $\langle\cdot\rangle\,\cdot$, the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Henceforth, x denotes the variable $x = (\hat{x}, z)$ and for any function φ, we denote $(\nabla_s\varphi)^{tr} = (\nabla_{\hat{y}}\varphi, 0)^{tr}$.

We define the solution space W by

$$W := \left\{ v \in L^2(0,T;H_0^1(\Omega)) : \frac{dv}{dt} \in L^2(0,T;H^{-1}(\Omega)) \right\}.$$

Definition 4.1. A random process U_ε with values in W is called a *weak* solution of (4.3) if, for almost all $\omega \in \mathcal{O}$, it satisfies the following integral identity:

(a) for each $\varphi \in H_0^1(\Omega)$,

$$\langle \frac{dU_\varepsilon}{dt}, \eta_{\omega,\varepsilon}\varphi \rangle + \int_\Omega A_\varepsilon \nabla U_\varepsilon \cdot \nabla \varphi \, dx - \int_\Omega z \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \frac{\partial U_\varepsilon}{\partial z} \varphi \, dx = \int_\Omega \eta_{\omega,\varepsilon} f_\varepsilon \varphi \, dx \qquad \mathbb{P}\text{-a.s,}$$

(b) $U_\varepsilon(\cdot, 0) = h_\varepsilon(\cdot)$ in $L^2(\Omega)$.

The following result ensures the existence and uniqueness of the problem (4.3).

Theorem 4.1. *Let η, η_1 satisfies the hypothesis (H1) and τ be an $(n-1)$ dimensional ergodic dynamical system on $\mathcal{O}$. Then, for almost all $\omega \in \mathcal{O}$, there exists a unique weak solution U_ε of (4.3) in the sense of Definition 4.1. Furthermore, almost surely for all $\varepsilon > 0$ the following uniform estimates hold:*

$$\|U_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|U_\varepsilon\|_W \le C \left\{ \|f_\varepsilon\|_{L^2(Q^T)} + \|h_\varepsilon\|_{L^2(\Omega)} \right\}, \tag{4.4}$$

for some constant $C > 0$ (independent of ε and ω), and hence,

$$\mathbb{E}\|U_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \mathbb{E}\|U_\varepsilon\|_W \le \sqrt{2} C \left\{ \|f\|_{L^2(\hat{Q}^T)} + \|h\|_{L^2(\hat{\Omega}^0)} \right\}, \tag{4.5}$$

for a constant $C > 0$ (independent of ε).

Sketch of Proof. For the readers convenience, we provide a short sketch of the proof (specially the uniform estimate 4.4) to the above theorem. In particular, using $\varphi = U_\varepsilon$ as a test function in Definition 4.1(a), we get

$$\langle \frac{dU_\varepsilon}{dt}, \ \eta_{\omega,\varepsilon} U_\varepsilon \ \rangle + \int_\Omega A_\varepsilon \nabla U_\varepsilon \cdot \nabla U_\varepsilon \, dx - \int_\Omega z \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \frac{\partial U_\varepsilon}{\partial z} U_\varepsilon \, dx = \int_\Omega \eta_{\omega,\varepsilon} f_\varepsilon U_\varepsilon \, dx \qquad \mathbb{P}\text{-a.s,}$$

Thanks to the hypothesis (H1), the following identity holds:

$$\frac{d}{dt} \int_\Omega |U_\varepsilon|^2 \eta_{\omega,\varepsilon} \, dx = 2 \left\langle \frac{dU_\varepsilon}{dt}, \eta_{\omega,\varepsilon} U_\varepsilon \right\rangle + \int_\Omega |U_\varepsilon|^2 \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \, dx.$$

Using this identity, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |U_\varepsilon|^2 \eta_{\omega,\varepsilon} \, dx + \int_\Omega A_\varepsilon \nabla U_\varepsilon \cdot \nabla U_\varepsilon \, dx - \int_\Omega z \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \frac{\partial U_\varepsilon}{\partial z} U_\varepsilon \, dx - \frac{1}{2} \int_\Omega |U_\varepsilon|^2 \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \, dx = \int_\Omega \eta_{\omega,\varepsilon} f_\varepsilon U_\varepsilon \, dx.$$

To get a priory estimates, note that from the uniform ellipticity of A_ε, we have the following inequality:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega |U_\varepsilon(t)|^2 \eta_{\omega,\varepsilon} \, dx + \alpha \|\nabla U_\varepsilon\|^2_{L^2(\Omega)^n} &\le \int_\Omega \eta_{\omega,\varepsilon} |f_\varepsilon U_\varepsilon| \, dx + \|\frac{\partial \eta_{\omega,\varepsilon}}{\partial t}\|_{L^\infty(Q^T)} \int_\Omega |\frac{\partial U_\varepsilon}{\partial z} U_\varepsilon| \, dx \\ &+ \|\frac{\partial \eta_{\omega,\varepsilon}}{\partial t}\|_{L^\infty(Q^T)} \int_\Omega |U_\varepsilon|^2 \, dx. \end{aligned}$$

For any $\delta > 0$, we know that $ab \leq \frac{1}{2\delta}a^2 + \frac{\delta}{2}b^2$. Choosing $a = \frac{1}{\sqrt{2}}\,|U_\varepsilon|\,,\; b = \sqrt{2}\left|\frac{\partial U_\varepsilon}{\partial z}\right|$, we obtain

$$\begin{aligned}\int_\Omega \left|\frac{\partial U_\varepsilon}{\partial z}\right| |U_\varepsilon|\; dx &\leq \frac{\delta}{2}\int_\Omega 2\left|\frac{\partial U_\varepsilon}{\partial z}\right|^2 dx + \frac{1}{2\delta}\int_\Omega \frac{1}{2}|U_\varepsilon|^2\, dx \\ &= \delta\int_\Omega \left|\frac{\partial U_\varepsilon}{\partial z}\right|^2 dx + \frac{1}{4\delta}\int_\Omega |U_\varepsilon|^2\, dx.\end{aligned}$$

Let $C^* := \left\|\frac{\partial \boldsymbol{\eta}}{\partial t}\right\|_{L^\infty(\mathcal{O}\times(0,T))} + \left\|\frac{\partial \boldsymbol{\eta}_1}{\partial t}\right\|_{L^\infty(\mathcal{O}\times(0,T))}$. Using the facts $\frac{1}{2} \leq \eta_{\omega,\varepsilon} \leq 2$ and $\left\|\frac{\partial \eta_{\omega,\varepsilon}}{\partial t}\right\|_{L^\infty(Q^T)} \leq C^*$ along with the above inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\,U_\varepsilon(t)\,|^2\eta_{\omega,\varepsilon}\, dx + \alpha\|\nabla U_\varepsilon\|^2_{L^2(\Omega)^n} &\leq 2\|f_\varepsilon(t)\|_{L^2(\Omega)}\|U_\varepsilon(t)\|_{L^2(\Omega)} + C^*\delta\int_\Omega |\frac{\partial U_\varepsilon}{\partial z}|^2\, dx \\ &\quad + \frac{C^*}{4\delta}\int_\Omega |U_\varepsilon|^2\, dx + C^*\int_\Omega |\,U_\varepsilon\,|^2\, dx \\ &\leq 2C_{\Omega,n}\|f_\varepsilon(t)\|_{L^2(\Omega)}\|\nabla U_\varepsilon(t)\|_{L^2(\Omega)^n} + C^*\delta\|\nabla U_\varepsilon(t)\|^2_{L^2(\Omega)^n} \\ &\quad + 2C^*1 + \frac{2C^*}{4\delta}\int_\Omega |U_\varepsilon|^2\eta_{\omega,\varepsilon}\, dx,\end{aligned}$$

where $C_{\Omega,n}$ is Poincaré constant. Again using Young's inequality with the chosen $\delta > 0$, for the choice $a = \sqrt{2}C_{\Omega,n}\|f_\varepsilon(t)\|_{L^2(\Omega)}$ and $b = \sqrt{2}\|\nabla U_\varepsilon(t)\|_{L^2(\Omega)^n}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\,U_\varepsilon(t)\,|^2\eta_{\omega,\varepsilon}\, dx + \alpha\,\|\nabla U_\varepsilon\|^2_{L^2(\Omega)^n} &\leq \frac{C^2_{\Omega,n}}{\delta}\,\|f_\varepsilon(t)\|^2_{L^2(\Omega)} + \delta\,\|\nabla U_\varepsilon(t)\|^2_{L^2(\Omega)^n} + C^*\delta\,\|\nabla U_\varepsilon(t)\|^2_{L^2(\Omega)^n} \\ &\quad + \gamma\int_\Omega |U_\varepsilon|^2\eta_{\omega,\varepsilon}\, dx \\ &\leq \frac{C^2_{\Omega,n}}{\delta}\,\|f_\varepsilon(t)\|^2_{L^2(\Omega)} + \delta\max(1, C^*)\,\|\nabla U_\varepsilon(t)\|^2_{L^2(\Omega)^n} \\ &\quad + \gamma\int_\Omega |U_\varepsilon|^2\eta_{\omega,\varepsilon}\, dx,\end{aligned}$$

where $\gamma := 2C^*\left(1 + \frac{2C^*}{4\delta}\right)$. We must find $\delta > 0$ such that $\delta\max(1, C^*) \leq \alpha/2$. It is easy to see that the inequality is valid for the choice $\delta = \frac{\alpha}{2\max(1,C^*)}$. Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\,U_\varepsilon t|^2\eta_{\omega,\varepsilon}\, dx + \frac{\alpha}{2}\|\nabla U_\varepsilon\|^2_{L^2(\Omega)^n} \leq \frac{C^2_{\Omega,n}}{\delta}\|f_\varepsilon(t)\|^2_{L^2(\Omega)} + \gamma\int_\Omega |U_\varepsilon|^2\eta_{\omega,\varepsilon}\, dx.$$

Equivalently,

$$\frac{1}{2}\frac{d}{dt}[e^{-2\gamma t}\int_\Omega |U_\varepsilon(t)|^2\eta_{\omega,\varepsilon}\, dx] + \frac{\alpha}{2}e^{-2\gamma t}\|U_\varepsilon\|^2_{H^1_0(\Omega)} \leq \frac{e^{-2\gamma t}C^2_{\Omega,n}}{\delta}\|f_\varepsilon(t)\|^2_{L^2(\Omega)^n}.$$

Integrating the above inequality over $(0, s)$ for $s \in (0, T)$, using the initial condition $U_\varepsilon(\cdot, 0) = h_\varepsilon$, we obtain

$$\frac{e^{-2\gamma s}}{2}\int_\Omega |U_\varepsilon(s)|^2\eta_{\omega,\varepsilon}\, dx + \frac{\alpha}{2}\int_0^s e^{-2\gamma t}\,\|U_\varepsilon\|^2_{H^1_0(\Omega)}\, dt \leq \int_0^s \frac{e^{-2\gamma t}C^2_{\Omega,n}}{\delta}\,\|f_\varepsilon(t)\|^2_{L^2(\Omega)}\, dt + \frac{1}{2}\int_\Omega |h_\varepsilon|^2\eta_{\omega,\varepsilon}(0)\, dx.$$

Taking supremum over $s \in (0,T)$ on both sides, we get

$$\begin{aligned}\frac{1}{2}\|U_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha e^{-2\gamma T}}{2}\|U_\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))} &\leq \frac{1}{2}\|U_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\alpha}{2}\int_0^T e^{-2\gamma s}\|U_\varepsilon\|^2_{H_0^1(\Omega)}\,ds \\ &\leq \frac{C^2_{\Omega,n}}{\delta}\int_0^T \|f_\varepsilon(t)\|^2_{L^2(\Omega)}\,dt + \|h_\varepsilon\|^2_{L^2(\Omega)},\end{aligned}$$

where we have used the fact that $e^{-2\gamma t}$ is strictly decreasing on $[0.T]$ for the chosen constant $\gamma \geq 0$. Note that the constants $\delta > 0, \gamma \geq 0$, and $C_{\Omega,n}$ are independent of both $\varepsilon > 0$ and $\omega \in \mathcal{O}$. Therefore, there exists a constant $C > 0$ (depends on α, δ, γ and Poincaré constant $C_{\Omega,n}$) such that U_ε satisfies the estimate (4.4). Recall that the noncylindrical domain $\hat{Q}^T$ is independent $\omega \in \mathcal{O}$ and $Q_\varepsilon^T \subset \hat{Q}^T$, and hence,

$$\begin{aligned}\|f_\varepsilon\|^2_{L^2(Q^T)} = \int_{Q^T} |f(\hat{x}, z\eta_{\omega,\varepsilon}, t)|^2\,d\hat{x}\,dz\,dt &= \int_{Q_\varepsilon^T} \frac{1}{\eta_{\omega,\varepsilon}} |f(\hat{x}, x_n, t)|^2\,d\hat{x}\,dx_n\,dt \\ &\leq 2\int_{\hat{Q}^T} |f(\hat{x}, x_n, t)|^2\,d\hat{x}\,dx_n\,dt = 2\,\|f\|^2_{L^2(\hat{Q}^T)}.\end{aligned}$$

Since $\Omega_\varepsilon^0 \subset \hat{\Omega}^0$, similar arguments in appropriate spaces yield $\|h_\varepsilon\|_{L^2(\Omega)} \leq \sqrt{2}\|h\|_{L^2(\hat{\Omega}^0)}$. Using these estimates in (4.4), we get

$$\|U_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|U_\varepsilon\|_W \leq \sqrt{2}C\{\|f\|_{L^2(\hat{Q}^T)} + \|h\|_{L^2(\hat{\Omega}^0)}\}.$$

The above bound holds uniformly in $\varepsilon > 0$ and almost all $\omega \in \mathcal{O}$. Now, taking the expectation on both sides, we obtain the desired estimate (4.5).

The existence of solution follows from the standard Galerkin method for a parabolic PDE. Following the lines of the above estimates, we can obtain the estimates in appropriate spaces for approximate solutions.

The convergence of the data is needed to prove the homogenization result and presented in the following two lemmas.

Lemma 4.1. *For $k \in \mathbb{Z}$, the following statements are true: for almost all $\omega \in \mathcal{O}$*

(i) $\eta^k_{\omega,\varepsilon}$ converges to η^k_ω in the uniform topology of $C_b(\mathbb{T}^{n-1} \times (0,T))$;
(ii) If $v_\varepsilon \to v$ strongly (resp. weakly) in $L^2(Q^T)$, then $v_\varepsilon \eta^k_{\omega,\varepsilon} \to v\eta^k_\omega$ strongly (resp. weakly) in $L^2(Q^T)$.

The proof of the above lemma is a simple consequence of hypothesis (H1) and is stated here only for completeness sake. Using the coordinate transformation corresponding to η_ω and the convergence obtained in the above lemma, we show the convergences of the data f_ε and h_ε in the following lemma. The proof of the next result is exactly as in Muthukumar et al,[1] except that we consider the random coordinate transformation here. For the completeness, we provide the proof.

Lemma 4.2. *Let $\mathcal{T}_{\eta_\omega} : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n \times [0,T)$ be defined by $\mathcal{T}_{\eta_\omega}(\hat{x}, z, t) = (\hat{x}, z\eta_\omega(\hat{x}, t), t)$. Define the functions $f_\eta : Q^T \to \mathbb{R}$ and $h_\eta : \Omega \to \mathbb{R}$ by*

$$f_\eta(\hat{x}, z, t) := f \circ \mathcal{T}_{\eta_\omega}(\hat{x}, z, t) \text{ and } \quad h_\eta := h \circ \mathcal{T}_{\eta_\omega}(\hat{x}, z, 0).$$

Then for almost all $\omega \in \mathcal{O}$,

(i) $f_\varepsilon \to f_\eta$ strongly in $L^2(Q^T)$;
(ii) $h_\varepsilon \to h_\eta$ strongly in $L^2(\Omega)$.

Proof.

(i) Recall that $f \in L^2(\hat{Q}^T)$. By the density of $C(\hat{Q}^T)$ in $L^2(\hat{Q}^T)$, for any $\delta > 0$, there exists a $\phi \in C(\hat{Q}^T)$ such that $\|f - \phi\|_{L^2(\hat{Q}^T)} \leq \frac{\delta}{3}$. For each $\omega \in \mathcal{O}$, we define $\phi_\varepsilon, \phi_{\eta_\omega} : Q^T \to \mathbb{R}$ by $\phi_\varepsilon(\hat{x}, z, t) := \phi \circ \mathcal{T}_{\omega,\varepsilon}(\hat{x}, z, t)$ and $\phi_{\eta_\omega}(\hat{x}, z, t) :=$

$\phi \circ \mathcal{T}_{\eta_\omega}(\hat{x}, z, t)$. Then,

$$\begin{aligned}\|f_\varepsilon - \phi_\varepsilon\|_{L^2(Q^T)} &= \left(\int_{Q^T} |f(\hat{x}, z\eta_{\omega,\varepsilon}, t) - \phi(\hat{x}, z\eta_{\omega,\varepsilon}, t)|^2\, d\hat{x}\, dz\, dt\right)^{\frac{1}{2}} \\ &= \left(\int_{Q^T_\varepsilon} \frac{1}{\eta_{\omega,\varepsilon}} |f(\hat{x}, x_n, t) - \phi(\hat{x}, x_n, t)|^2\, d\hat{x}\, dx_n\, dt\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\hat{Q}^T} \frac{1}{\eta_{\omega,\varepsilon}} |f(\hat{x}, x_n, t) - \phi(\hat{x}, x_n, t)|^2\, d\hat{x}\, dx_n\, dt\right)^{\frac{1}{2}} \\ &\leq \frac{\delta}{3} \left\| \frac{1}{\eta_{\omega,\varepsilon}} \right\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^{n-1}\times(0,T))}.\end{aligned}$$

Arguing similarly, we also obtain $\|\phi_{\eta_\omega} - f_\eta\|_{L^2(Q^T)} \leq \frac{\delta}{3} \left\|\frac{1}{\eta_\omega}\right\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^{n-1}\times(0,T))}$. Now, by the continuity of ϕ, for our chosen $\delta > 0$, there exists a $\tau_\delta > 0$ such that $|\phi(\hat{x}, z\eta_{\omega,\varepsilon}(\hat{x}, t), t) - \phi(\hat{x}, z\eta_\omega(\hat{x}, t), t)| \leq \frac{\delta}{3}$ whenever $|z\eta_{\omega,\varepsilon}(\hat{x}, t) - z\eta_\omega(\hat{x}, t)| < \tau_\delta$. By the uniform convergence of $z\eta_{\omega,\varepsilon}$ to $z\eta_\omega$ in $\Omega \times (0, T)$ (q.v. Lemma 4.1i), there exists a $N \in \mathbb{N}$ such that $|z\eta_{\omega,\varepsilon}(\hat{x}, t) - z\eta_\omega(\hat{x}, t)| < \tau_\delta$, for all $\varepsilon \leq 1/N$. Thus, for all $\varepsilon \leq 1/N$,

$$|\phi(\hat{x}, z\eta_{\omega,\varepsilon}(\hat{x}, t), t) - \phi(\hat{x}, z\eta_\omega(\hat{x}, t), t)| \leq \frac{\delta}{3}.$$

Hence, for all $\varepsilon \leq 1/N$, we have

$$\|\phi_\varepsilon - \phi_{\eta_\omega}\|_{L^2(Q^T)} = \left(\int_{Q^T} |\phi(\hat{x}, z\eta_{\omega,\varepsilon}, t) - \phi(\hat{x}, z\eta_\omega, t)|^2\, d\hat{x}\, dz\, dt\right)^{\frac{1}{2}} \leq |Q^T|^{\frac{1}{2}} \frac{\delta}{3}.$$

Therefore, for any given $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $\varepsilon \leq 1/N$,

$$\begin{aligned}\|f_\varepsilon - f_\eta\|_{L^2(Q^T)} &\leq \|f_\varepsilon - \phi_\varepsilon\|_{L^2(Q^T)} + \|\phi_\varepsilon - \phi_{\eta_\omega}\|_{L^2(Q^T)} + \|\phi_{\eta_\omega} - f_\eta\|_{L^2(Q^T)} \\ &\leq \max\left(\left\|\frac{1}{\eta_{\omega,\varepsilon}}\right\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^{n-1}\times(0,T))}, \left\|\frac{1}{\eta_\omega}\right\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^{n-1}\times(0,T))}, |Q^T|^{\frac{1}{2}} \right) \delta.\end{aligned}$$

Thus, we have the desired strong convergence of f_ε to f_η.

(ii) Using arguments similar to (i) above in the appropriate function space we prove that, for any arbitrary $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $\varepsilon \leq \frac{1}{N}$,

$$\|h_\varepsilon - h_\eta\|_{L^2(\Omega))} \leq \max\left(\left\|\frac{1}{\eta_{\omega,\varepsilon}(0)}\right\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^{n-1})}, \left\|\frac{1}{\eta_\omega(0)}\right\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^{n-1})}, |\Omega|^{\frac{1}{2}} \right) \delta.$$

Thus, we obtain the desired strong convergence of $h_\varepsilon \to h_\eta$ in $L^2(\Omega)$.

□

5 | HOMOGENIZATION

In this section, we study the homogenization (4.3). To this end, we have to list down the convergence of the solution U_ε of (4.3) in various function spaces and then those converges, we derive the effective equation.

Theorem 5.1. *Let $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_1$ satisfy (H1). If, for each $\varepsilon > 0$, U_ε is the weak solution of (4.3), then for almost all $\omega \in \mathcal{O}$, there exist $U_0 \in W$ such that*

(a) $U_\varepsilon \rightharpoonup U_0$ weakly in W.

(b) $U_\varepsilon \rightharpoonup U_0$ *weak-* in* $L^\infty(0,T;L^2(\Omega))$.
(c) $U_\varepsilon \to U_0$ *strongly in* $L^2(Q^T)$.
(d) $U_\varepsilon \to U_0$ *strongly in* $C([0,T];H^{-1}(\Omega))$.
(e) $A_\varepsilon \nabla U_\varepsilon \to A^\star \nabla U_0$ *weakly in* $L^2\left(0,T;L^2(\Omega)^n\right)$,

where U_0 *is the unique weak solution of*

$$\begin{cases} \eta_\omega \frac{\partial U_0}{\partial t} - div\left(A^\star \nabla U_0\right) - z\frac{\partial \eta_\omega}{\partial t}\frac{\partial U_0}{\partial z} = \eta_\omega f_\eta & in \quad Q^T, \\ U_0 = 0 & on\, (\Gamma_0 \cup \Gamma_1) \times (0,T), \\ U_0(\hat{x}, z, 0) = h_\eta & in \quad \Omega. \end{cases} \tag{5.1}$$

Here, the limit coefficient $A^\star$ *is given by*

$$A^\star(\hat{x}, z, t, \omega) := \begin{pmatrix} \eta_\omega(\hat{x},t) I_{n-1} & -z\nabla_{\hat{x}}\eta_\omega(\hat{x},t) \\ -z\nabla_{\hat{x}}\eta_\omega(\hat{x},t)^{tr} & \frac{1+z^2\,|\,\nabla_{\hat{x}}\eta_\omega\,|^2}{\eta_\omega} \end{pmatrix}. \tag{5.2}$$

Proof. From the estimate (4.4), the sequence of weak solutions $\{U_\varepsilon\}$ is uniformly bounded in both W and $L^\infty(0,T;L^2(\Omega))$. Hence, there exist a $U_0 \in W \cap L^\infty(0,T;L^2(\Omega))$ and a subsequence of ε (not relabeled) such that the weak convergence (a) and the weak-* convergence (b) are satisfied. Since W is compactly embedded in $L^2(Q^T)$ (Aubin–Lions–Simon lemma), we get the strong convergence (c) for the same subsequence. Again using the Aubin–Lions–Simon lemma for the case $p=\infty$, the space

$$\{v \in L^\infty(0,T;L^2(\Omega))\,|\,\frac{dv}{dt} \in L^2(0,T;H^{-1}(\Omega))\}$$

is compactly embedded in $C([0,T];H^{-1}(\Omega))$. Hence, for the same subsequence, the convergence (d) holds true.

Further, $\|A_\varepsilon \nabla U_\varepsilon\|_{L^2(0,T;L^2(\Omega)^n)} \le C$, and hence, again there exist a subsequence of ε (not relabeled) and a $\xi_0^\omega \in L^2\left(0,T;L^2(\Omega)^n\right)$ such that $A_\varepsilon \nabla U_\varepsilon \rightharpoonup \xi_0^\omega$ (up to subsequence) weakly in $L^2\left(0,T;L^2(\Omega)^n\right)$. Thus, for every $v \in H_0^1(\Omega)$,

$$\begin{aligned} \int_\Omega \eta_\omega f_\eta v\,dx = \lim_{\varepsilon\to 0}\int_\Omega \eta_{\omega,\varepsilon} f_\varepsilon v\,dx &= \lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon \nabla U_\varepsilon \cdot \nabla v\,dx - \lim_{\varepsilon\to 0}\int_\Omega z\frac{\partial \eta_{\omega,\varepsilon}}{\partial t}\frac{\partial U_\varepsilon}{\partial z}\varphi\,dx \\ &= \int_\Omega \xi_0^\omega \cdot \nabla v\,dx - \int_\Omega z\frac{\partial \eta_\omega}{\partial t}\frac{\partial U_0 \partial z}{v}\,dx, \end{aligned} \tag{5.3}$$

for almost all $\omega \in \mathcal{O}$. In the last limit of the first line, we have used the weak convergence of $\partial U_\varepsilon/\partial z$ and the strong convergence of $\partial \eta_{\omega,\varepsilon}/\partial t$. Our main job is to identify ξ_0^ω in terms of U_0 such that (5.3) is satisfied. To do this, we define the functions $w_j(x) = x \cdot \boldsymbol{e}_j$ for $1 \le j \le n$, and hence, $\nabla w_j = \boldsymbol{e}_j$. For each $1 \le j \le n-1$,

$$A_\varepsilon \boldsymbol{e}_j = \left(\eta_{\omega,\varepsilon}\boldsymbol{e}_j - z\frac{\partial \eta_\omega}{\partial x_j}\boldsymbol{e}_n - z\frac{\partial \eta_{1,\omega}}{\partial y_j}\left(\frac{\hat{x}}{\varepsilon}, t\right)\boldsymbol{e}_n\right)^{tr} \text{ and } \nabla_{\hat{y}}\eta_1\left(\frac{\hat{x}}{\varepsilon}, t, \omega\right) \rightharpoonup^* \mathcal{M}_{\nabla_{\hat{y}}\eta_{1,\omega}}(t) = \boldsymbol{0}_{n-1},$$

we get $A_\varepsilon \boldsymbol{e}_j \rightharpoonup^* A^\star \boldsymbol{e}_j$ weak-* in $L^2\left(0,T;L^\infty(\Omega)^n\right)$. For each $1 \le j \le n-1$ and $v \in H_0^1(\Omega)$,

$$\begin{aligned} \int_\Omega A_\varepsilon \nabla w_j \cdot \nabla v\,dx = \int_\Omega A_\varepsilon \boldsymbol{e}_j \cdot \nabla v\,dx &= \int_\Omega \eta_{\omega,\varepsilon}\frac{\partial v}{\partial x_j}\,dx - \int_\Omega z\frac{\partial \eta_{\omega,\varepsilon}}{\partial x_j}\frac{\partial v}{\partial z}\,dx \\ &= -\int_\Omega \frac{\partial \eta_{\omega,\varepsilon}}{\partial x_j}v\,dx + \int_\Omega \frac{\partial \eta_{\omega,\varepsilon}}{\partial x_j}v\,dx \\ &= 0, \end{aligned} \tag{5.4}$$

where the second last equality followed by integration by parts and the fact that φ vanishes on the boundary $\Gamma_0 \cup \Gamma_1$. Using the weak-* convergence $A_\varepsilon \boldsymbol{e}_j \rightharpoonup A^* \boldsymbol{e}_j$ in $L^2(0,T;L^\infty(\Omega)^n)$, we get

$$\int_\Omega A^* \boldsymbol{e}_j \cdot \nabla v \, dx = 0 \qquad \forall v \in H_0^1(\Omega). \tag{5.5}$$

In particular, using $v = U_\varepsilon \varphi$ for arbitrary $v \in C_c^\infty(\Omega)$ as a test function in (5.4), we get

$$0 = \int_\Omega (A_\varepsilon \boldsymbol{e}_j \cdot \nabla U_\varepsilon)\varphi \, dx + \int_\Omega (A_\varepsilon \boldsymbol{e}_j \cdot \nabla \varphi) U_\varepsilon \, dx.$$

Since A_ε is symmetric and $U_\varepsilon \to U_0$ strongly in $L^2(Q^T)$ (q.v.(c)), passing the limit in the above equation to get

$$\int_\Omega (\boldsymbol{e}_j \cdot \xi_0)\varphi \, dx = -\int_\Omega (A^* \boldsymbol{e}_j \cdot \nabla \varphi) U_0 \, dx = \int_\Omega (A^* \boldsymbol{e}_j \cdot \nabla U_0)\varphi \, dx,$$

where the last equality follows from (5.5) with test function $v = U_0 \varphi$. Therefore, we obtain $\xi_0 \cdot \boldsymbol{e}_j = A^* \nabla U_0 \cdot \boldsymbol{e}_j$ for $1 \le j \le n-1$.

It remains to show that $\xi_0 \cdot \boldsymbol{e}_n = A^* \nabla U_0 \cdot \boldsymbol{e}_n$. For each $w_j(x) = x \cdot \boldsymbol{e}_j$, using $v = \eta_\omega \varphi w_j$ as test function in (5.3), we obtain

$$\int_\Omega \eta_\omega^2 f_\eta \varphi w_j \, dx = \int_\Omega [\xi_0 \cdot \nabla(\eta_\omega \varphi)] w_j \, dx + \int_\Omega [\xi_0 \cdot \boldsymbol{e}_j] \eta_\omega \varphi \, dx - \int_\Omega z \frac{\partial \eta_\omega}{\partial t} \frac{\partial U_0}{\partial z} \eta_\omega \varphi w_j \, dx. \tag{5.6}$$

Taking $v_\varepsilon = A_\varepsilon \nabla U_\varepsilon \cdot \boldsymbol{e}_i$ in Lemma 4.1 (ii) yields $\eta_{\omega,\varepsilon} A_\varepsilon \nabla U_\varepsilon \rightharpoonup \eta_\omega \xi_0$ in $L^2\left(0,T;L^2(\Omega)^n\right)$. Thus, using $v = \eta_{\omega,\varepsilon}\varphi w_j$ as test function in Definition 4.1(a),

$$\begin{aligned}
\int_\Omega \eta_{\omega,\varepsilon}^2 f_\varepsilon \varphi w_j \, dx = & \int_\Omega [\eta_{\omega,\varepsilon} A_\varepsilon \nabla U_\varepsilon \cdot \boldsymbol{e}_j]\varphi \, dx + \int_\Omega [A_\varepsilon \nabla U_\varepsilon \cdot \nabla(\eta_\omega \varphi)] w_j \, dx \\
& + \int_\Omega \left[A_\varepsilon \nabla U_\varepsilon \cdot \nabla_s \eta_{1,\omega}\left(\frac{\hat{x}}{\varepsilon}, t\right)\right] w_j \varphi \, dx \\
& + \varepsilon \int_\Omega [A_\varepsilon \nabla U_\varepsilon \cdot \nabla \varphi]\, \eta_{1,\omega}\left(\frac{\hat{x}}{\varepsilon}, t\right) w_j \, dx - \int_\Omega z \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \frac{\partial U_\varepsilon}{\partial z} \eta_{\omega,\varepsilon} \varphi w_j \, dx.
\end{aligned}$$

Taking both side limit as $\varepsilon \to 0$ and using weak convergence of $\eta_{\omega,\varepsilon} A_\varepsilon \nabla U_\varepsilon$ and $A_\varepsilon \nabla U_\varepsilon$, we get

$$\begin{aligned}
\int_\Omega \eta_\omega^2 f_\eta \varphi w_j \, dx = & \int_\Omega [\eta_\omega \xi_0 \cdot \boldsymbol{e}_j]\varphi \, dx + \int_\Omega [\xi_0 \cdot \nabla(\eta_\omega \varphi)] w_j \, dx \\
& + \lim_{\varepsilon \to 0} \int_\Omega \left[A_\varepsilon \nabla U_\varepsilon \cdot \nabla_s \eta_1\left(\frac{\hat{x}}{\varepsilon}, \omega, t\right)\right] w_j \varphi \, dx - \int_\Omega z \frac{\partial \eta_\omega}{\partial t} \frac{\partial U_0}{\partial z} \eta_\omega \varphi w_j \, dx.
\end{aligned}$$

Using the identity (5.6) and the above equality, we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega \left[A_\varepsilon \nabla U_\varepsilon \cdot \nabla_s \eta_1\left(\frac{\hat{x}}{\varepsilon}, \omega, t\right)\right] w_j \varphi \, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega \left[A_\varepsilon \nabla U_\varepsilon \cdot \nabla_s \eta_1\left(\frac{\hat{x}}{\varepsilon}, \omega, t\right)\right] w_j \varphi \, dx = 0 \; \forall \varphi \in L^2(\Omega). \tag{5.7}$$

The nth component of $A_\varepsilon \nabla U_\varepsilon$ denoted as $\xi_\varepsilon^n = A_\varepsilon \nabla U_\varepsilon \cdot \boldsymbol{e}_n$, and hence, $\eta_{\omega,\varepsilon}\xi_\varepsilon^n \rightharpoonup \eta_\omega \xi_0^n$ weakly in $L^2(Q^T)$. Moreover, $\eta_{\omega,\varepsilon}\xi_\varepsilon^n$ can be written as

$$\eta_{\omega,\varepsilon}\xi_\varepsilon^n = \nabla U_\varepsilon \cdot \eta_\omega A^* \boldsymbol{e}_n - \varepsilon z \eta_{1,\omega} \nabla_{\hat{x}} U_\varepsilon \cdot \nabla_{\hat{x}} \eta_\omega + z^2 \frac{\partial U_\varepsilon}{\partial z} \nabla_{\hat{x}} \eta_\omega \cdot \nabla_{\hat{y}} \eta_1 \left(\frac{\hat{x}}{\varepsilon}, \omega, t \right) + z A_\varepsilon \nabla U_\varepsilon \cdot \nabla_s \eta_1 \left(\frac{\hat{x}}{\varepsilon}, \omega, t \right).$$

For any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega \eta_\omega \xi_0^n \varphi \, dx &= \lim_{\varepsilon \to 0} \int_\Omega \eta_{\omega,\varepsilon} \xi_\varepsilon^n \varphi \, dx \\ &= \lim_{\varepsilon\to 0} \int_\Omega \left[\nabla U_\varepsilon \cdot \eta_\omega A^* \boldsymbol{e}_n \right] \varphi \, dx + \lim_{\varepsilon \to 0} \varepsilon \int_\Omega \left[-z\eta_1 \nabla_{\hat{x}} U_\varepsilon \cdot \nabla_{\hat{x}} \eta_\omega \right] \varphi \, dx \\ &+ \lim_{\varepsilon\to 0} \int_\Omega z^2 \frac{\partial U_\varepsilon}{\partial z} \left[\nabla_{\hat{x}} \eta_\omega \cdot \nabla_{\hat{y}} \eta_1 \left(\frac{\hat{x}}{\varepsilon}, \omega, t \right) \right] \varphi \, dx \\ &+ \lim_{\varepsilon\to 0} \int_\Omega A_\varepsilon \nabla U_\varepsilon \cdot \nabla_s \eta_1 \left(\frac{\hat{x}}{\varepsilon}, \omega, t \right) z\varphi \, dx.\end{aligned}$$

Second term on the right hand side is zero, and from (5.7), the last term on the right hand side is also zero for the case $j = n$. Further, using $\eta_\omega \varphi A^* \boldsymbol{e}_n \in L^2\left(0, T; L^2(\Omega)^n\right)$ as test function for weak convergence of ∇U_ε and integration by parts in third integral, we have

$$\int_\Omega \eta_\omega \xi_0^n \varphi \, dx = \int_\Omega (\nabla U_0 \cdot \eta_\omega A^* \boldsymbol{e}_n) \varphi \, dx - \lim_{\varepsilon\to 0} \int_\Omega U_\varepsilon \left[\nabla_{\hat{x}} \eta_\omega \cdot \nabla_{\hat{y}} \eta_1 \left(\frac{\hat{x}}{\varepsilon}, \omega, t \right) \right] \frac{\partial (z^2 \varphi)}{\partial z} dx.$$

From the strong convergence of U_ε, it is easy to see that $\frac{\partial(z^2\varphi)}{\partial z} \frac{\partial \eta_\omega}{\partial x_i} U_\varepsilon \to \frac{\partial(z^2\varphi)}{\partial z} \frac{\partial \eta_\omega}{\partial x_i} U_0$ strongly in $L^2(Q^T)$ and the fact that $\frac{\partial \eta_1}{\partial y_i} \rightharpoonup 0$ weakly in $L^2(Q^T)$ (Birkhoff Ergodic theorem); thus, last term on RHS above identity is equal to zero. Hence,

$$\int_\Omega \eta_\omega \xi_0^n \varphi \, dx = \int_\Omega (\eta_\omega A^* \nabla U_0 \cdot \boldsymbol{e}_n) \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega).$$

Therefore, we have identified ξ_0^n in terms of U_0, and it is given by $\xi_0^n = A^* \nabla U_0 \cdot \boldsymbol{e}_n$. Hence, for almost all $\omega \in \mathcal{O}$, we have

$$A_\varepsilon \nabla U_\varepsilon \rightharpoonup A^\star \nabla U_0 \text{ in } L^2\left(0, T; L^2(\Omega)^n\right). \tag{5.8}$$

Now, we identified $\xi_0 = A^\star \nabla U_0$, and hence, the integral identity (5.3) becomes

$$\left\langle \frac{\partial U_0}{\partial t}, \eta_\omega \varphi \right\rangle + \int_\Omega A^\star \nabla U_0 \cdot \nabla \varphi \, dx - \int_\Omega z \frac{\partial \eta_\omega}{\partial t} \frac{\partial U_0}{\partial z} \phi \, dx = \int_\Omega \eta_\omega f_\eta \varphi \, dx, \tag{5.9}$$

for all $\varphi \in H_0^1(\Omega)$. The above equation is nothing but the weak formulation of (5.1).

Next, we want to show that $U_0(\cdot, 0) = h(\cdot)$ in $L^2(\Omega)$. For this, we choose $\varphi \in H_0^1(\Omega)$ and $\psi \in C^\infty([0, T])$ such that $\psi(0) = 1$ and $\psi(T) = 0$. Multiplying ψ on both side of Definition 4.1(a) with φ as test function and integrate over the time variable, we obtain

$$\begin{aligned}\int_{Q^T} f_\varepsilon \varphi \psi \, dx dt + \psi(0) \int_\Omega h_\varepsilon \varphi \, dx &= - \int_{Q^T} \frac{d\psi}{dt} \varphi U_\varepsilon \, dx dt + \int_{Q^T} (A_\varepsilon \nabla U_\varepsilon \cdot \nabla \varphi) \psi \, dx dt \\ &- \int_{Q^T} z \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} \frac{\partial U_\varepsilon}{\partial z} \varphi \psi \, dx dt,\end{aligned}$$

where we used the integration by parts for functions in W with respect to the time variable. Taking limit, as $\varepsilon \to 0$, both sides and using the convergences proved in the previous step, we get

$$\int_{Q^T} f_\eta \varphi \psi \, dx dt + \int_\Omega h_\eta \varphi \, dx = -\int_{Q^T} \frac{d\psi}{dt} \varphi U_0 \, dx dt + \int_{Q^T} \left(A^\star \nabla U_0 \cdot \nabla \varphi \right) \psi \, dx dt$$
$$- \int_{Q^T} z \frac{\partial \eta_\omega}{\partial t} \frac{\partial U_0}{\partial z} \phi \psi \, dx dt.$$

By the integration by parts with respect to the t variable, we obtain

$$\int_{Q^T} f_\eta \varphi \psi \, dx dt + \int_\Omega h_\eta \varphi \, dx = \int_0^T \left\langle \frac{dU_0}{dt}, \varphi \right\rangle \psi \, dt - \int_\Omega U_0(x,T)\varphi(x)\psi(T) \, dx$$
$$+ \int_\Omega U_0(x,0)\varphi(x)\psi(0) \, dx + \int_{Q^T} \left(A^\star \nabla U_0 \cdot \nabla \varphi \right) \psi \, dx dt$$
$$- \int_{Q^T} z \frac{\partial \eta_\omega}{\partial t} \frac{\partial U_0}{\partial z} \phi \psi \, dx dt..$$

Multiply our choice of ψ both side of (5.9) and integrate over $(0, T)$, we get

$$\int_\Omega U_0(x,0)\varphi(x) \, dx = \int_\Omega h_\eta(x)\varphi(x) \, dx, \ \forall \varphi \in H_0^1(\Omega),$$

where we have used the fact that ψ satisfies $\psi(0) = 1$ and $\psi(T) = 0$. Thus, $U_0(\cdot, 0) = h_\eta(\cdot)$ in $L^2(\Omega)$ by the density of $H_0^1(\Omega)$ in $L^2(\Omega)$.

To complete the proof, it remains to show that $A^\star$ is uniformly elliptic. Again using the Schur complement of $\eta_\omega I_{n-1}$ in $A^\star$ and Aitken block-diagonalisation formula, one can show that (the followings lines below 4.3 for A_ε replaced by $A^\star$) there exists positive constants α^* and βa^* such that

$$\alpha^2 |\xi|^2 \le A^\star(\hat{x}, z, t, \omega)\xi \cdot \xi, \qquad \left| A^\star(\hat{x}, z, t, \omega)\xi \right| \le \beta a^* |\xi| \qquad \text{a.e. } \forall \xi \in \mathbb{R}^n. \tag{5.10}$$

Further, due to the uniqueness of solution of (5.1), the convergences in Theorem 5.1(a)–(e) are valid for the entire sequence. □

Remark 1. We emphasize that the solution U_0 identified as the solution of u with the coordinate transformation $\mathcal{T}_{\eta_\omega}$,

$$\begin{cases} \frac{\partial u}{\partial t}(\hat{x}, x_n, t, \omega) - \Delta u(\hat{x}, x_n, t, \omega) = f(\hat{x}, x_n, t) & \text{in } Q^T_{\eta_\omega}, \\ u(\hat{x}, \eta_\omega(\hat{x}, t), t, \omega) = 0 & \text{on } \Sigma^T_{\eta_\omega} \cup \Gamma_0 \times (0, T), \\ u(\hat{x}, x_n, 0, \omega) = h(\hat{x}, x_n) & \text{in } \Omega^0_{\eta_\omega}, \end{cases}$$

where $Q^T_{\eta_\omega}, \Omega^t_{\eta_\omega}, \Gamma^t_{\eta_\omega}, \Sigma^T_{\eta_\omega}$ are defined by replacing $\eta_{\omega,\varepsilon}$ with η_ω in Section 2. Notice that this problem also depends on an event $\omega \in \mathcal{O}$. In particular, we can choose η_ω to be a deterministic function (q.v. Figure 2), and hence, the limit problem is deterministic.

6 | CORRECTOR RESULTS

We know that problems with homogeneous Dirichlet boundary condition on time independent oscillating boundary, the strong convergence between solutions holds true in homogenization process. It is not clear that whether U_ε strongly converges to U_0 in $L^2(0, T; H_0^1(\Omega))$ in our setting. This is the subject of this section, and we need some more technical result to prove the strong convergence. Recall that $A_\varepsilon \nabla U_\varepsilon \rightharpoonup A^\star \nabla U_0$ in $L^2\left(0, T; L^2(\Omega)^n\right)$ and the strong convergence of $\{U_\varepsilon\}$ in $C([0, T]; H^{-1}(\Omega))$. While proving corrector result, we also establish the strong convergence of $\{U_\varepsilon\}$ in $C([0, T]; L^2(\Omega))$. To understand the corrector term, for any $\varphi \in H_0^1(\Omega)$ (in particular U_0), we need to find the limit of $A_\varepsilon \nabla \varphi$. The matrix A_ε can be decomposed as

$$A_\varepsilon = \begin{pmatrix} \eta_\omega I_{n-1} & -z\nabla_{\hat{x}}\eta_\omega(\hat{x},t) \\ -z\nabla_{\hat{x}}\eta_\omega(\hat{x},t)^{tr} & \frac{1+z^2|\nabla_{\hat{x}}\eta_\omega|^2}{\eta_{\omega,\varepsilon}} \end{pmatrix} + \begin{pmatrix} \varepsilon\eta_{1,\omega,\varepsilon} I_{n-1} & -z\nabla_{\hat{y}}\eta_1\left(\frac{\hat{x}}{\varepsilon},t,\omega\right) \\ -z\nabla_{\hat{y}}\eta_1\left(\frac{\hat{x}}{\varepsilon},t,\omega\right)^{tr} & \frac{z^2\nabla_{\hat{x}}\eta\cdot\nabla_{\hat{y}}\eta_1\left(\frac{\hat{x}}{\varepsilon},t,\omega\right)+z^2|\nabla_{\hat{y}}\eta_1|^2\left(\frac{\hat{x}}{\varepsilon},t,\omega\right)}{\eta_{\omega,\varepsilon}} \end{pmatrix}.$$

Again invoking Birkhoff ergodic theorem for the entries of A_ε along with Lemma 4.1, we get

$$A_\varepsilon \nabla\varphi \rightharpoonup A^\star \nabla\varphi + \frac{z^2}{\eta}\frac{\partial\varphi}{\partial z}\|\nabla_{\hat{y}}\eta_{1,\omega}\|^2_{L^2(\mathbb{T}^{n-1})}\boldsymbol{e}_n. \tag{6.1}$$

Lemma 6.1. *For any given $k \in \mathbb{N}$, let $\{\Phi_\varepsilon\} \subset L^2(0,T;L^2(\Omega)^k)$ and $\{\Psi_\varepsilon\} \subset L^\infty(0,T;L^2(\Omega)^k)$ be uniformly bounded sequences in their respective spaces and, for each $t \in [0,T]$, set $\Theta_\varepsilon(t) := \int_0^t \int_\Omega \Phi_\varepsilon(x,\tau)\cdot\Psi_\varepsilon(x,\tau)\,dx\,d\tau$. If $\Theta_\varepsilon(t) \to \Theta_0(t)$ converges to some $\Theta_0 \in C[0,T]$ pointwise, then $\Theta_\varepsilon \to \Theta_0$ in $C[0,T]$.*

The proof of this lemma can be found in Muthukumar et al[1] and which has been proved by using the Arzelà–Ascoli result.

Corollary 6.1. *Let U_ε and U_0 be weak solutions of (4.3) and (5.1), respectively. For each $t \in [0,T]$, define*

$$\Theta_{1,\varepsilon}(t) := \int_0^t \int_\Omega \eta_{\omega,\varepsilon} f_\varepsilon(x,s)U_\varepsilon(x,s)\,dx\,ds;\ \ \Theta_1(t) := \int_0^t \int_\Omega \eta_\omega f_\eta(x,s)U_0(x,s)\,dx\,ds,$$

and

$$\Theta_{2,\varepsilon}(t) := \int_0^t \int_\Omega \frac{\partial\eta_{\omega,\varepsilon}}{\partial t}U_\varepsilon\left[\frac{U_\varepsilon}{2} + z\frac{\partial U_\varepsilon}{\partial z}\right]dx\,ds;\ \ \Theta_2(t) := \int_0^t \int_\Omega \frac{\partial\eta_\omega}{\partial t}U_0\left[\frac{U_0}{2} + z\frac{\partial U_0}{\partial z}\right]dx\,ds.$$

Then for almost all $\omega \in \mathcal{O}$,

a. $\Theta_{1,\varepsilon} \to \Theta_1$ in $C([0,T])$, and
b. $\Theta_{2,\varepsilon} \to \Theta_2$ in $C([0,T])$

Proof.

(a) From Lemma 4.2(i) and Theorem 5.1(b), the functions $\Phi_\varepsilon = f_\varepsilon$ and $\Psi_\varepsilon = \eta_{\omega,\varepsilon}U_\varepsilon$ satisfy the hypotheses of Lemma 6.1 for $k = 1$. Thus, $\Theta_{1,\varepsilon} \to \Theta_1$ uniformly in $C([0,T])$.

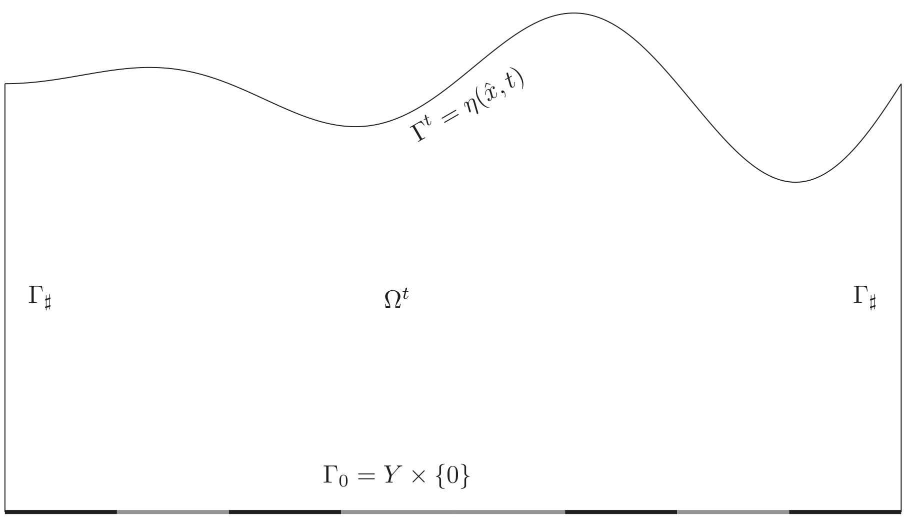

FIGURE 2 Slice of the evolving domain with boundary ($\boldsymbol{\eta} = 6 + \frac{\hat{x}}{8}\sin(2\pi\hat{x})$) at time t

(b) The functions $\Phi_\varepsilon = \frac{U_\varepsilon}{2} + z\frac{\partial U_\varepsilon}{\partial z}$ and $\Psi_\varepsilon = \frac{\partial \eta_{\omega,\varepsilon}}{\partial t} U_\varepsilon$ satisfy the hypotheses of Lemma 6.1 for $k = 1$, because $\frac{\partial \eta_{\omega,\varepsilon}}{\partial t} U_\varepsilon$ and $\frac{U_\varepsilon}{2} + z\frac{\partial U_\varepsilon}{\partial z}$ converges to $\frac{\partial \eta_\omega}{\partial t} U_0$ and $\frac{U_0}{2} + z\frac{\partial U_0}{\partial z}$, strongly and weakly, respectively. Hence, we conclude that $\Theta_{2,\varepsilon} \to \Theta_2$ in $C([0,T])$.

□

Let us define the following energy functions associated with the systems (4.3) and (5.1).

$$\begin{cases} i) \quad E_\varepsilon(t) := \frac{1}{2}\int_\Omega |U_\varepsilon(x,t)|^2 \eta_{\omega,\varepsilon}\, dx + \int_0^t \int_\Omega A_\varepsilon(x,s)\nabla U_\varepsilon(x,s) \cdot \nabla U_\varepsilon(x,s)\, dx\, ds \\ ii) \quad E(t) := \frac{1}{2}\int_\Omega |U_0(x,t)|^2 \eta_\omega\, dx + \int_0^t \int_\Omega A^\star(x,s)\nabla U_0(x,s) \cdot \nabla U_0(x,s)\, dx\, ds. \end{cases} \tag{6.2}$$

We have the following theorem.

Theorem 6.1. *Assume that both $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_1$ satisfy the hypothesis (H1). Let U_ε and U_0 be the weak solutions of (4.3) and (5.1), respectively. Then for almost all $\omega \in \mathcal{O}$, $E_\varepsilon \to E$ in C[0, T].*

Proof. For any $v \in W$ and $\psi \in W^{1,\infty}(\mathbb{T}^{n-1} \times (0,T))$, we have the following identity:

$$\frac{d}{dt}\int_\Omega |v|^2 \psi\, dx = 2\left\langle \frac{dv}{dt}, \psi v \right\rangle + \int_\Omega |v|^2 \frac{\partial \psi}{\partial t}\, dx.$$

In particular, the above identity holds true for both the cases $v = U_\varepsilon, \psi = \eta_{\omega,\varepsilon}$, and $v = U_0, \psi = \eta$. Using U_ε and U_0 as a test function in Definition 4.1(a) and (5.9), respectively, along with the above identity, and then integrating over $Q^t := \Omega \times (0,t)$, we obtain

$$E_\varepsilon(t) = \frac{1}{2}\int_\Omega |h_\varepsilon|^2 \eta_{\omega,\varepsilon}(\hat{x},0)\, dx + \Theta_{1,\varepsilon}(t) - \Theta_{2,\varepsilon}(t)$$

and

$$E(t) = \frac{1}{2}\int_\Omega |h_\eta|^2 \eta_\omega(\hat{x},0)\, dx + \Theta_1(t) - \Theta_2(t).$$

From Corollary 6.1, it is enough to show that the first term on the right hand side of E_ε converges to that of E because both are independent of time variable t. By Lemma 4.2(ii) and Lemma 4.1(ii), we get $\frac{1}{2}\int_\Omega |h_\varepsilon|^2 \eta_{\omega,\varepsilon}(\hat{x},0)\, dx \to \frac{1}{2}\int_\Omega |h_\eta|^2 \eta_\omega(\hat{x},0)\, dx$ in $C([0,T])$. □

Lemma 6.2. *Let us assume that $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_1$ satisfy (H1). Also, let U_ε and U_0 be the unique weak solution of (4.3) and (5.1), respectively. For any $\phi \in C^\infty([0,T]; C_c^\infty(\Omega))$, for almost all $\omega \in \mathcal{O}$, we define*

$$\mathcal{F}^\varepsilon_{\phi,\omega}(x,t) := \frac{z}{\eta(\hat{x},\omega,t)} \frac{\partial \phi}{\partial z} \nabla_s \eta_1\left(\frac{\hat{x}}{\varepsilon}, t, \omega\right).$$

Further, for each $t \in [0,T]$, let

$$I^\varepsilon_{\phi,\omega}(t) := \frac{1}{2}\int_\Omega |U_\varepsilon(t) - \phi(t)|^2 \eta_{\omega,\varepsilon}\, dx + \int_0^t \int_\Omega A_\varepsilon\left(\nabla U_\varepsilon - \nabla\phi - \mathcal{F}^\varepsilon_{\phi,\omega}\right) \cdot \left(\nabla U_\varepsilon - \nabla\phi - \mathcal{F}^\varepsilon_{\phi,\omega}\right) dx\, ds$$

and

$$I_{\phi,\omega}(t) := \frac{1}{2}\int_\Omega |U_0(t) - \phi(t)|^2 \eta_\omega\, dx + \int_0^t \int_\Omega A^\star\,(\nabla U_0 - \nabla\phi) \cdot (\nabla U_0 - \nabla\phi)\, dx\, ds.$$

Then for almost all $\omega \in \mathcal{O}$, $I^\varepsilon_{\phi,\omega}$ converges to $I_{\phi,\omega}$ in C[0, T].

Proof. The corrector term is motivated by the convergence (6.1) and by the following observation:

$$
\begin{aligned}
A_\varepsilon \mathcal{F}^\varepsilon_{\phi,\omega} &= \left(\frac{z\eta_{\omega,\varepsilon}}{\eta_\omega} \frac{\partial \varphi}{\partial z} \nabla_{\hat{y}} \eta_{1,\omega} \left(\frac{\hat{x}}{\varepsilon}, t \right), \frac{-z^2}{\eta_\omega} \frac{\partial \varphi}{\partial z} \right)^{tr} \\
&\rightharpoonup -\frac{z^2}{\eta_\omega} \frac{\partial \varphi}{\partial z} \|\nabla_{\hat{y}} \eta_{1,\omega}\|^2_{L^2(\mathbb{T}^{n-1})} \boldsymbol{e}_n \qquad \text{weakly in} \quad L^2\left(0,T; L^2(\Omega)^n\right),
\end{aligned}
\tag{6.3}
$$

where we have used (3.1) and the Birkhoff ergodic theorem for $|\nabla_{\hat{y}} \eta_{1,\omega}|^2 \left(\frac{\hat{x}}{\varepsilon}, t\right)$. Note that $I^\varepsilon_{\phi,\omega}$ can rewritten as

$$
I^\varepsilon_{\phi,\omega}(t) = E_\varepsilon(t) + \frac{1}{2}\int_\Omega |\phi(t)|^2 \eta_{\omega,\varepsilon}\, dx - \int_\Omega U_\varepsilon(t)\phi(t)\eta_{\omega,\varepsilon}\, dx + \sigma_1^\varepsilon(t) - \sigma_2^\varepsilon(t),
$$

where E_ε is defined in (6.2) and

$$
\begin{cases}
\sigma_1^\varepsilon(t) &= \displaystyle\int_0^t \int_\Omega A_\varepsilon \left(\nabla\phi + \mathcal{F}^\varepsilon_{\phi,\omega} \right) \cdot \left(\nabla\phi + \mathcal{F}^\varepsilon_{\phi,\omega} \right) dx\, ds \\
\sigma_2^\varepsilon(t) &= 2\displaystyle\int_0^t \int_\Omega A_\varepsilon \nabla U_\varepsilon \cdot \left(\nabla\phi + \mathcal{F}^\varepsilon_{\phi,\omega} \right) dx\, ds
\end{cases}.
$$

By Theorem 6.1, E_ε uniformly converges to E in $[0,T]$, where $E(t)$ is defined as in (6.2). For each $t \in [0,T]$,

$$
\frac{1}{2} \left| \int_\Omega |\phi(t)|^2 \eta_{\omega,\varepsilon}\, dx - \int_\Omega |\phi(t)|^2 \eta_\omega\, dx \right| \le \frac{\varepsilon \|\eta_{1,\omega}\|_\infty}{2} \int_\Omega |\phi(t)|^2\, dx
$$

and

$$
\begin{aligned}
\left| \int_\Omega U_\varepsilon(t)\phi(t)\eta_{\omega,\varepsilon}\, dx - \int_\Omega U_0(t)\phi(t)\eta_\omega\, dx \right| &\le \|\eta_\omega\|_\infty \left| \int_\Omega U_\varepsilon(t)\phi(t)\, dx - \int_\Omega U_0(t)\phi(t)\, dx \right| \\
&\quad + \varepsilon \|\eta_{1,\omega}\|_\infty \left| \int_\Omega U_\varepsilon(t)\phi(t)\, dx \right| \\
&\le \|\eta_\omega\|_\infty \|U_\varepsilon(t) - U_0(t)\|_{H^{-1}(\Omega)} \|\phi(t)\|_{H^1_0(\Omega)} \\
&\quad + \varepsilon \|\eta_{1,\omega}\|_\infty \|U_\varepsilon(t)\|_{L^2(\Omega)} \|\phi(t)\|_{L^2(\Omega)}.
\end{aligned}
$$

Taking supremum over $[0,T]$ of the above two inequalities and then pass to the limit $\varepsilon \to 0$, we obtain

$$
\lim_{\varepsilon\to 0} \frac{1}{2} \left\| \int_\Omega |\phi(t)|^2 \eta_{\omega,\varepsilon}\, dx - \int_\Omega |\phi(t)|^2 \eta_\omega\, dx \right\|_{C[0,T]} = 0 \tag{6.4}
$$

and

$$
\lim_{\varepsilon\to 0} \left\| \int_\Omega U_\varepsilon \phi \eta_{\omega,\varepsilon}\, dx - \int_\Omega U_0 \phi \eta_\omega\, dx \right\|_{C[0,T]} \le \|\eta_\omega\|_\infty \|\phi\|_{C([0,T];H^1_0(\Omega))} \lim_{\varepsilon\to 0} \|U_\varepsilon - U_0\|_{C([0,T];H^{-1}(\Omega))}.
$$

From Theorem 5.1(d) $U_\varepsilon \to U_0$ in $C([0,T]; H^{-1}(\Omega))$, we have

$$
\lim_{\varepsilon\to 0} \left\| \int_\Omega U_\varepsilon \phi \eta_{\omega,\varepsilon}\, dx - \int_\Omega U_0 \phi \eta_\omega\, dx \right\|_{C[0,T]} = 0. \tag{6.5}
$$

Next we shall show that the uniform convergence of $\{\sigma_1^\varepsilon\}$. For each fixed $t \in [0,T]$,

$$
\lim_{\varepsilon\to 0} \sigma_1^\varepsilon(t) = \lim_{\varepsilon\to 0} \int_0^t \int_\Omega A_\varepsilon \nabla\phi \cdot \nabla\phi\, dx\, ds + 2\lim_{\varepsilon\to 0} \int_0^t \int_\Omega A_\varepsilon \mathcal{F}^\varepsilon_{\phi,\omega} \cdot \nabla\phi\, dx\, ds + \lim_{\varepsilon\to 0} \int_0^t \int_\Omega |\mathcal{F}^\varepsilon_{\phi,\omega}|^2\, dx\, ds.
$$

Using the weak convergence (6.1) of $A_\varepsilon \nabla \varphi$ in the first term on the right hand side and the weak convergence (6.3) of $A_\varepsilon \mathcal{F}^\varepsilon_{\phi,\omega}$ in the second term on the right hand side, we obtain

$$\begin{aligned}\sigma_1(t) :=& \lim_{\varepsilon\to 0}\sigma_1^\varepsilon(t)\\ =& \int_0^t\int_\Omega \left[A^\star\nabla\phi + \frac{z^2}{\eta}\frac{\partial\phi}{\partial z}\|\nabla_{\hat{y}}\eta_{1,\omega}\|^2_{L^2(\mathbb{T}^{n-1})}\boldsymbol{e}_n\right]\cdot\nabla\phi\,dx\,ds\\ &-2\int_0^t\int_\Omega\left[\frac{z^2}{\eta}\frac{\partial\phi}{\partial z}\|\nabla_{\hat{y}}\eta_{1,\omega}\|^2_{L^2(\mathbb{T}^{n-1})}\boldsymbol{e}_n\right]\cdot\nabla\phi\,dx\,ds + \lim_{\varepsilon\to 0}\int_0^t\int_\Omega |\mathcal{F}^\varepsilon_{\phi,\omega}|^2\,dx\,ds.\end{aligned}$$

Again invoking the Birkhoff ergodic theorem for $|\nabla_{\hat{y}}\eta_1|^2(\hat{x}/\varepsilon, t,\omega)$, limit of the last term can be identified as

$$\int_0^t\int_\Omega\frac{z^2}{\eta}\left(\frac{\partial\phi}{\partial z}\right)^2\|\nabla_{\hat{y}}\eta_{1,\omega}\|^2_{L^2(\mathbb{T}^{n-1})}\,dx\,ds = \int_0^t\int_\Omega\left[\frac{z^2}{\eta}\frac{\partial\phi}{\partial z}\|\nabla_{\hat{y}}\eta_{1,\omega}\|^2_{L^2(\mathbb{T}^{n-1})}\boldsymbol{e}_n\right]\cdot\nabla\varphi\,dx\,ds.$$

Thus, $\lim_{\varepsilon\to 0}\sigma_1^\varepsilon(t) = \sigma_1(t) = \int_0^t\int_\Omega A^\star\nabla\phi\cdot\nabla\phi$. For the choice of $\Phi_\varepsilon = A_\varepsilon\left(\nabla\phi + \mathcal{F}^\varepsilon_{\phi,\omega}\right)$ and $\Psi_\varepsilon = \nabla\phi + \mathcal{F}^\varepsilon_{\phi,\omega}$ with above pointwise limit satisfy the hypotheses of Lemma 6.1 with $k=n$, we obtain the uniform convergence

$$\lim_{\varepsilon\to 0}\|\sigma_1^\varepsilon - \sigma_1\|_{C[0,T]} = 0. \tag{6.6}$$

Finally, using (5.7) for $j = n$, $\varphi = \partial\phi/\partial z$ and the weak convergence of $A_\varepsilon\nabla U_\varepsilon$ (q.v. Theorem 5.1e), for each fixed $t\in[0,T]$:

$$\begin{aligned}\sigma_2(t) := \lim_{\varepsilon\to 0}\sigma_2^\varepsilon(t) &= \lim_{\varepsilon\to 0}\int_0^t\int_\Omega A_\varepsilon\nabla U_\varepsilon\cdot\nabla\phi\,dx\,ds + \lim_{\varepsilon\to 0}\int_0^t\int_\Omega A_\varepsilon\nabla U_\varepsilon\cdot\mathcal{F}^\varepsilon_{\phi,\omega}\,dx\,ds\\ &= \int_0^t\int_\Omega A^\star\nabla U_0\cdot\nabla\phi\,dx\,ds.\end{aligned}\tag{6.7}$$

Again invoking Lemma 6.1 for the choice $\Phi_\varepsilon = A_\varepsilon\nabla U_\varepsilon$ and $\Psi_\varepsilon = \nabla\phi + \mathcal{F}^\varepsilon_{\phi,\omega}$ with $k=n$, we get

$$\lim_{\varepsilon\to 0}\|\sigma_2^\varepsilon - \sigma_2\|_{C[0,T]} = 0. \tag{6.8}$$

Using the convergences (6.4)–(6.8) and taking into account of Theorem 6.1, we conclude that $I^\varepsilon_{\phi,\omega}$ uniformly converges to the sum of the limits obtained in (6.4)–(6.8). Moreover, the sum is exactly $I_{\phi,\omega}$, and hence, we obtain the uniform convergence of $I^\varepsilon_{\phi,\omega}\to I_{\phi,\omega}$ in $[0,T]$. □

Now, we prove an important result which is the key step to obtain the corrector result.

Lemma 6.3. *Assume that* $\boldsymbol{\eta}$ *and* $\boldsymbol{\eta}_1$ *satisfy the hypothesis (H1). Let* U_ε *be the unique solution of (4.3). Then, almost all* $\omega\in\mathcal{O}$ *it holds that for every* $\delta>0$*, there exist a* $\phi_\delta\in C^\infty([0,T];C_c^\infty(\Omega))$*, a positive constant C and* $N_\delta\in\mathbb{N}$ *such that*

$$\frac{1}{4}\|U_\varepsilon - \phi_\delta\|^2_{C([0,T];L^2(\Omega))} + \alpha\left\|\nabla U_\varepsilon - \nabla\phi_\delta - \mathcal{F}^\varepsilon_{\phi_\delta,\omega}\right\|^2_{L^2(0,T;[L^2(\Omega)]^n)} \le C\delta^2 \text{ for all } \varepsilon\le\frac{1}{N_\delta},$$

where $\alpha>0$ *is the ellipticity constant of* A_ε *and* $\mathcal{F}^\varepsilon_{\phi_\delta,\omega}(x,t,\omega) := \frac{z}{\eta_\omega}\frac{\partial\phi_\delta}{\partial z}\nabla_s\eta_1\left(\frac{\hat{x}}{\varepsilon},t,\omega\right)$.

Proof. According to Proposition 3.60 on Cioranescu and Donato,[9] the space $C^\infty([0,T];C_c^\infty(\Omega))$ is dense in $L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$. For any given $\delta>0$, there exists $\phi_\delta\in C^\infty([0,T];C_c^\infty(\Omega))$ such that

$$\|U_0 - \phi_\delta\|_{C([0,T];L^2(\Omega))}\le\delta \text{ and } \|\nabla U_0 - \nabla\phi_\delta\|_{L^2(0,T;[L^2(\Omega)]^n)}\le\frac{\delta}{2\sqrt{\beta^\star}}, \tag{6.9}$$

where $\beta^\star$ is uniform bound of $A^\star$ and is given in (5.10).

Using uniform lower bound of $\eta_{\omega,\varepsilon}$ and the uniform ellipticity of A_ε in $I^\varepsilon_{\phi_\delta,\omega}$, for each $t \in [0,T]$, we obtain

$$\frac{1}{4}\|U_\varepsilon(t)-\phi_\delta(t)\|^2_{L^2(\Omega)}+\alpha\int_0^t\left\|\nabla U_\varepsilon-\nabla\phi_\delta-\mathcal{F}^\varepsilon_{\phi_\delta,\omega}\right\|^2_{[L^2(\Omega)]^n}ds\le I^\varepsilon_{\phi_\delta,\omega}(t),$$

where $I^\varepsilon_{\phi_\delta,\omega}(t)$ is defined in Lemma 6.2 for the choice $\phi=\phi_\delta$. Taking supremum over $[0,T]$ on both sides of above inequality, we get

$$\begin{aligned}\alpha\left\|\nabla U_\varepsilon-\nabla\phi_\delta-\mathcal{F}^\varepsilon_{\phi_\delta,\omega}\right\|^2_{L^2(0,T;L^2(\Omega)^n)}&\\+\frac{1}{4}\|U_\varepsilon-\phi_\delta\|^2_{C([0,T];L^{2(\Omega)})}&\le\left\|I^\varepsilon_{\phi_\delta,\omega}\right\|_{C[0,T]}\\&\le\|I^\varepsilon_{\phi_\delta,\omega}-I^\delta\|_{C[0,T]}+\|I^\delta\|_{C[0,T]},\end{aligned}\tag{6.10}$$

where $I_{\phi_\delta,\omega}(t)$ is defined in Lemma 6.2 for the choice $\phi=\phi_\delta$. From the uniform convergence of $I^\varepsilon_{\phi_\delta,\omega}$ to $I_{\phi_\delta,\omega}$ in $[0,T]$ (q.v. Lemma 6.2 for $\phi=\phi_\delta$), for the chosen $\delta>0$, there exists a $N_\delta\in\mathbb{N}$ such that, for all $\varepsilon\le\frac{1}{N_\delta}$,

$$\left\|I^\varepsilon_{\phi_\delta,\omega}-I_{\phi_\delta,\omega}\right\|_{C[0,T]}\le\delta^2.\tag{6.11}$$

Now, we estimate the last term $I_{\phi_\delta,\omega}$ on the RHS of (6.10). Using uniform upper bound of $\eta_{\omega,\varepsilon}$ and the uniform bound of $A^\star$ in $I_{\phi_\delta,\omega}$, and then taking supremum over $[0,T]$, we get

$$\begin{aligned}\|I_{\phi_\delta,\omega}\|_{C[0,T])}&\le\|U_0-\phi_\delta\|^2_{C([0,T];L^{2(\Omega)})}+\beta^\star\|\nabla(U_0-\phi_\delta)\|^2_{L^20,T;L^2(\Omega)^n}\\&\le\delta^2+\beta^\star\delta^4\le2c_0\delta^2\end{aligned}\tag{6.12}$$

where $c_0:=\max\{1,\beta^\star\}$. Using the estimates (6.11) and (6.12) in (6.10), we obtain, for all $\varepsilon\le\frac{1}{N_\delta}$,

$$\frac{1}{4}\|U_\varepsilon-\phi_\delta\|^2_{C([0,T];L^{2(\Omega)})}+\alpha\left\|\nabla U_\varepsilon-\nabla\phi_\delta-\mathcal{F}^\varepsilon_{\phi_\delta,\omega}\right\|^2_{L^2(0,T;L^2(\Omega)^n)}\le C\delta^2$$

where $C:=\max\{1,2c_0\}$. □

The weak convergence of U_ε to U_0 in W implies its weak convergence in $L^2(0,T;H^1_0(\Omega))$ and, hence, ∇U_ε weakly converges to ∇U_0 in $L^2(0,T;L^2(\Omega)^n)$. The following corrector result gives the strong convergence of ∇U_ε with appropriate corrector.

Theorem 6.2. *Let us assume that both $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_O$ satisfy (H1). Let U_ε and U_0 be the unique solutions of (4.3) and (5.1), respectively. Then a.s (i.e., for almost all $\omega\in\mathcal{O}$)*

(a) $\lim_{\varepsilon\to0}\|U_\varepsilon-U_0\|_{C([0,T];L^2(\Omega))}=0$, *and*
(b) $\lim_{\varepsilon\to0}\left\|\nabla U_\varepsilon-\nabla U_0-\frac{z}{\eta_\omega}\frac{\partial U_0}{\partial z}\nabla_s\eta_1\left(\frac{\cdot}{\varepsilon},\cdot\right)\right\|_{L^2(0,T;L^2(\Omega)^n)}=0.$

Proof. For any given $\delta>0$, there exists $\phi_\delta\in C^\infty([0,T],C^\infty_c(\Omega)$ such that the estimate (6.9) is satisfied.

(a) From Lemma 6.3, for the chosen δ, there exists a constant $c_1>0$ and $N_\delta\in\mathbb{N}$ such that

$$\|U_\varepsilon-\phi_\delta\|_{C([0,T];L^2(\Omega))}\le c_1\delta\text{ for all }\quad\varepsilon\le\frac{1}{N_\delta}.$$

Using these two estimates, we obtain

$$\begin{aligned}\|U_\varepsilon-U_0\|_{C([0,T];L^2(\Omega))}&\le\|U_\varepsilon-\phi_\delta\|_{C([0,T];L^2(\Omega))}+\|\phi_\delta-U_0\|_{C([0,T];L^2(\Omega))}\\&\le c_1\delta+\delta=(c_1+1)\delta,\end{aligned}$$

for all $\varepsilon\le\frac{1}{N_\delta}$,

(b) Again by Lemma 6.3, for the chosen $\delta > 0$, the same ϕ_δ, positive constant C and $N_\delta \in \mathbb{N}$, we have

$$\alpha \left\| \nabla U_\varepsilon - \nabla \phi_\delta - \mathcal{F}^{\varepsilon}_{\phi_\delta,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)} \leq C\delta^2$$

for all $\varepsilon \leq \frac{1}{N_\delta}$. Let us set $\mathcal{F}^{\varepsilon}_{U_0,\omega}(x,t) = \frac{z}{\eta_\omega}\frac{\partial U_0}{\partial z}\nabla_{\hat{x}}\eta_{1,\omega}\left(\frac{\hat{x}}{\varepsilon},t\right)$. Adding and subtracting $\nabla\phi_\delta$ and $\mathcal{F}^{\varepsilon}_{\phi_\delta,\omega}$ to the limit in Theorem 6.2(b), and then using triangle inequality and Young's inequality, we get

$$\begin{aligned}\frac{1}{4}\left\| \nabla U_\varepsilon - \nabla U_0 - \mathcal{F}^{\varepsilon}_{U_0,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)} \leq & \left\| \nabla U_\varepsilon - \nabla \phi_\delta - \mathcal{F}^{\varepsilon}_{\phi_\delta,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)} \\ & + \|\nabla\phi_\delta - \nabla U_0\|^2_{L^2(0,T;L^2(\Omega)^n)} \\ & + \left\| \mathcal{F}^{\varepsilon}_{\phi_\delta,\omega} - \mathcal{F}^{\varepsilon}_{U_0,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)}.\end{aligned}$$

The second term in the above inequality can be bounded by using the density property (6.9) of the solution U_0 as $\|\nabla\phi_\delta - \nabla U_0\|^2_{L^2(0,T;L^2(\Omega)^n)} \leq \frac{\delta^2}{4\eta a^\star}$. Hence, we compute

$$\frac{1}{4}\left\| \nabla U_\varepsilon - \nabla U_0 - \mathcal{F}^{\varepsilon}_{U_0,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)} \leq \left(\frac{C}{\alpha} + \frac{1}{4\eta a^\star}\right)\delta^2 + \left\| \mathcal{F}^{\varepsilon}_{\phi_\delta,\omega} - \mathcal{F}^{\varepsilon}_{U_0,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)}. \tag{6.13}$$

As a consequence of (6.9), we have

$$\begin{aligned}\left\| \mathcal{F}^{\varepsilon}_{\phi_\delta,\omega} - \mathcal{F}^{\varepsilon}_{U_0,\omega} \right\|^2_{L^2(0,T;L^2(\Omega)^n)} \leq & \left\| \nabla_{\hat{x}}\eta_{1,\omega}\left(\frac{\cdot}{\varepsilon},\cdot\right) \right\|^2_{L^\infty(\mathbb{T}^{n-1}\times(0,T))} \left\| \frac{\partial\phi_\delta}{\partial z} - \frac{\partial U_0}{\partial z} \right\|^2_{L^2(0,T;L^2(\Omega))} \\ \leq & \frac{\delta^2}{4\eta a^\star}\|\nabla_{\hat{y}}\eta_{1,\omega}\|^2_{L^\infty(\mathbb{T}^{n-1}\times(0,T))}.\end{aligned}$$

Using this in the estimate (6.13), we conclude that there is a positive constant C_0 such that, for all $\varepsilon \leq \frac{1}{N_\delta}$,

$$\left\| \nabla U_\varepsilon - \nabla U_0 - \mathcal{F}^{\varepsilon}_{U_0,\omega} \right\|^2_{L^2(0,T;[L^2(\Omega)]^n)} \leq C_0\delta^2.$$

This completes the proof of the corrector result. □

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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